Ενότητα 2α

2. (a) $f(x) = x^2+2x+1$ $\quad \Delta=[-2,0]$

Η f συνεχής $[-2,0]$ ως π.σ.σ

Η f παρ/μη $(-2,0)$ ως π.π.σ.

$\left.\begin{array}{l} f(-2)=1 \\ f(0)=1 \end{array}\right\} f(-2)=f(0)$

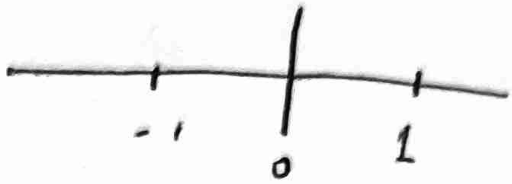
Rolle $\exists\, \xi \in (-2,0)$ τ.ω $f'(\xi)=0$

$f'(x) = 2x+2$

$f'(\xi)=0 \Rightarrow 2\xi+2=0$

$\xi=-1$

3. (a) $f(x) = \begin{cases} x^2, & x < 0 \\ x^3, & x \geq 0 \end{cases}$ $\quad \Delta = [-1,1]$



$\lim\limits_{x\to 0^-} f(x) = 0$, $\lim\limits_{x\to 0^+} f(x) = 0$ $\Big\}$ $\lim\limits_{x\to 0} f(x) = 0$

$f(0) = 0$ Συνεχής στο 0!

Επίσης η f συνεχής $[-1,0)$ και $(0,1]$ ως π.σ.σ ✓

Άρα συνεχής $[-1,1]$

$\lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \dfrac{x^2-0}{x-0} = \lim\limits_{x\to 0^-} x = 0$

$\lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \dfrac{x^3-0}{x} = 0$

$f'(0) = 0$ παραγωγίσιμη στο 0.

Η f παρ/μη $(-1,0)$ και $(0,1)$ ως π.π.δ

Άρα παρ/μη $(-1,1)$

$$\left.\begin{array}{l} f(-1)=1 \\ f(1)=1 \end{array}\right\}$$

Rolle $\exists \xi \in (-1,1)$ τ.ω $f'(\xi)=0$

$x<0$

$f'(\xi)=0$

$2\xi=0$

$\cancel{\xi=0}$

$x\geq 0$

$f'(\xi)=0$

$3\xi^2=0$

$\xi=0$

4. $f(x) = (x+1)\,\eta\mu x$

(α) $f(-1) = 0$, $f(0) = 0$ $\Big\}$ Rolle $\exists\, \xi \in (-1, 0)$ τ.ω $f'(\xi) = 0$.

(β) $\varepsilon\varphi x = -x - 1$ $\quad (-1, 0)$

$f'(x) = \eta\mu x + (x+1)\,\sigma\upsilon\nu x$

$f'(\xi) = \eta\mu \xi + (\xi+1)\,\sigma\upsilon\nu \xi = 0$

$\dfrac{\eta\mu\xi}{\sigma\upsilon\nu\xi} + \xi + 1 = 0$

$\varepsilon\varphi\xi + \xi + 1 = 0$

$\varepsilon\varphi\xi = -\xi - 1.$

5. $f(x) = x^4 - x^3 + \lambda x^2 - \lambda x$

(α) $\left.\begin{array}{l} f(0)=0 \\ f(1)=0 \end{array}\right\}$ Rolle $\exists \xi \in (0,1)$ τ.ω $f'(\xi)=0$.

(β) $f'(x) = 4x^3 - 3x^2 + 2\lambda x - \lambda$

$f'(\xi) = \boxed{4\xi^3 - 3\xi^2 + 2\lambda\xi - \lambda = 0}$ ✓

8. $g(x) = e^x f(x)$

$\left.\begin{array}{l} g(0) = e^0 f(0) = 0 \\ g(\frac{3}{2}) = e^{3/2} f(\frac{3}{2}) = 0 \end{array}\right\}$ $g(0) = g(\frac{3}{2})$ Rolle

$\exists \xi \in (0, \frac{3}{2})$ τ.ω

$g'(\xi) = 0$

$g'(x) = e^x f(x) + e^x f'(x)$

$g'(\xi) = e^\xi f(\xi) + e^\xi f'(\xi) = 0$

$f(\xi) + f'(\xi) = 0$

$f'(\xi) = -f(\xi)$

9. (α) $g(x) = \ln f(x) - \alpha x$ Rolle $[0, \alpha]$

$f(\alpha) = e f(0)$

$g(0) = \ln f(0)$

$g(\alpha) = \ln f(\alpha) - \alpha^2 = \ln e f(0) - \alpha^2 =$

$= \ln e + \ln f(0) - \alpha^2 =$

$= 1 + \ln f(0) - \alpha^2$

οπότε $g(0) = g(\alpha) \Leftrightarrow \cancel{\ln f(0)} = 1 + \cancel{\ln f(0)} - \alpha^2$

$\alpha^2 = 1$

$\alpha = 1$ $\cancel{\alpha = -1}$

(β) Αφού ισχύει ο Rolle

$\exists \xi \in (0, \alpha)$ τω $g'(\xi) = 0$

$$g'(x) = \frac{f'(x)}{f(x)} - \alpha$$

$$g'(\xi) = \frac{f'(\xi)}{f(\xi)} - 1 = 0 \Leftrightarrow f'(\xi) = f(\xi)$$

10. $f(x) = x^4 - 10x^3 - 15x^2 - x + 1$

ⓐ $\dfrac{10x^3 + 15x^2 + x}{x^4 + 1} = 1$ $\quad (-1, 1)$

$f(-1) = -2$

$f(0) = 1$

$f(1) = -14$

$f(-1)f(0) < 0$ Bolzano $\exists \xi_1 \in (-1, 0)$ τ.ω $f(\xi_1) = 0$

$f(0)f(1) < 0$ Bolzano $\exists \xi_2 \in (0, 1)$ τ.ω $f(\xi_2) = 0$.

ⓑ Αφού $f(\xi_1) = f(\xi_2)$ Rolle

$\exists \xi \in (\xi_1, \xi_2)$ τ.ω

$f'(\xi) = 0$.

$f'(x) = 4x^3 - 30x^2 - 30x - 1$

$f'(\xi) = 4\xi^3 - 30\xi^2 - 30\xi - 1 = 0$

12. Έστω ότι $g(x) = f(x) - x^3$ έχει 2 ρίζ-

$$g(x_1) = g(x_2) = 0$$

$$\underbrace{}_{\text{Rolle}}$$

$$g'(\xi) = 0.$$

$$g'(x) = f'(x) - 3x^2$$

$$g'(\xi) = f'(\xi) - 3\xi^2 = 0$$

$$f'(\xi) = 3\xi^2$$ Άτοπο!.

αφού $f'(x) \neq 3x^2$

14. (α) $g(0) = f(0) = 0$

$g(1) = f(1) - 1 = 0$ $\Big\}$ Rolle $\exists \xi \in (0,1)$ τ.ω. $g'(\xi) = 0$

(β) $g(x) = 0$

Έχω βρει ανά δύο ρίζες.

Έστω ότι έχει 3.

$$\underbrace{g(x_1) = g(x_2)}_{\text{Rolle}} \underbrace{= g(x_3)}_{\text{Rolle}} = 0$$

$$\underbrace{g'(\xi_1) = g'(\xi_2)}_{\text{Rolle}} = 0$$

$$g''(\xi) = 0.$$

(γ) Δύο κοινά σημεία $A(0,0)$ $B(1,1)$

$g'(x) = f'(x) - 2x$

$g''(x) = f''(x) - 2$

$g''(\xi) = f''(\xi) - 2 = 0 \Rightarrow f''(\xi) = 2$ Άτοπο!

Έχει το πολύ 2 ρίζες.

Έχω ανά 2 άρα έχει 2.

24. $f'(x) = \frac{f(x)}{x}$

β. $y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow (0,0)$

$0 - f(x_0) = f'(x_0)(0 - x_0)$

$-f(x_0) = -x_0 f'(x_0)$

$\frac{f(x_0)}{x_0} = f'(x_0)$

α. $g(x) = \frac{f(x)}{x}$

$g(2) = \frac{f(2)}{2} = \frac{f(3)}{3} = g(3)$ Rolle $\exists\, \xi \in (2,3)$

τ.ω $g'(\xi) = 0$

$g'(x) = \frac{f'(x)x - f(x)}{x^2}$

$g'(\xi) = \frac{f'(\xi)\xi - f(\xi)}{\xi^2} = 0$ $\quad f'(\xi) = \frac{f(\xi)}{\xi}$

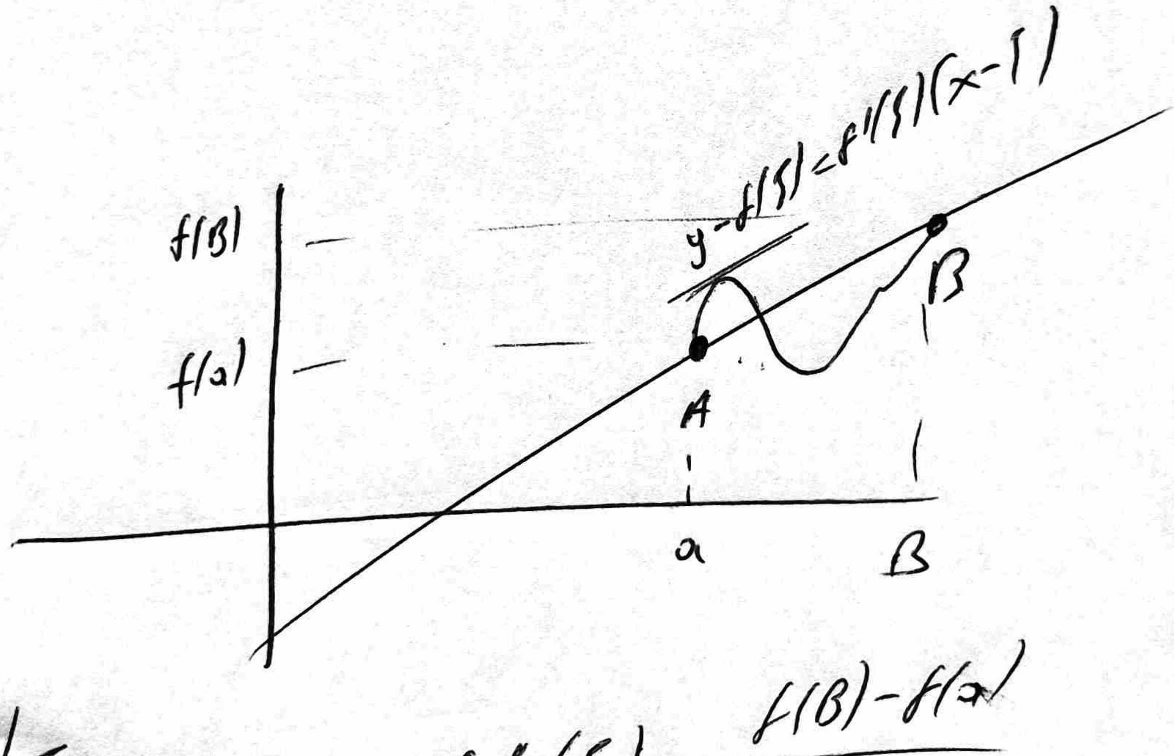
$f'(\xi)\xi = f(\xi)$

ΘΜΤ

f συνεχής $[\alpha, \beta]$

f παρ/μη (α, β)

$\exists \xi \in (\alpha, \beta)$ τ.ω $f'(\xi) = \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha}$.



$\varepsilon \parallel \zeta \Leftrightarrow \quad f'(\xi) = \dfrac{f(\beta) - f(\alpha)}{\beta - \alpha}$.

2. $f(x) = \sqrt{x+1}$, $[-1,3]$

Η συνεχής $[-1,3]$ ως π.σ.σ.

Η παρ/μη $(-1,3)$ ως π.π.σ.

Άρα $\exists \xi \in (-1,3)$ τ.ω $f'(\xi) = \dfrac{f(3)-f(-1)}{3-(-1)}$

$$f'(\xi) = \frac{2-0}{4} \Rightarrow f'(\xi) = \frac{1}{2}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

$$f'(\xi) = \frac{1}{2}$$

$$\frac{1}{2\sqrt{\xi+1}} = \frac{1}{2}$$

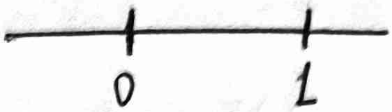
$$2\sqrt{\xi+1} = 2$$

$$\sqrt{\xi+1} = 1 \Rightarrow \xi+1 = 1$$

$$\xi = 0.$$

14. $f(x) = \begin{cases} x^3 + \alpha, & x<0 \\ 0, & x=0 \\ x^2, & x>0 \end{cases}$

(α) Για να είναι συνεχής στο $[0,1]$ πρέπει

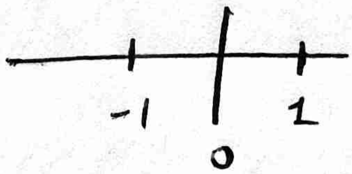


- συνεχής στο $(0,1)$ ✓
- $f(0) = 0$ $=\!=\!=$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$ ✓
- $f(1) = 1$ $=\!=\!=$ $\lim_{x\to 1^-} f(x) = 1$.

Συνεχής στο $[0,1]$

Επίσης παρ/μη $(0,1)$ και n.n.r.

(β) Απαιτώ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$



$\underline{\alpha = 0}$

① Αρκει νδο $\exists \xi \in (1,3)$ $f'(\xi) = \lambda_{AB}$

$$f'(\xi) = \frac{f(-1) - f(1)}{-1-1} = \frac{-1-1}{-2} = 1$$

$$f'(\xi) = 1$$

$$y - f(\xi) = f'(\xi)(x-\xi) \parallel \varepsilon_{AB}$$

$$f'(\xi) = \lambda_{AB} = \frac{f(1) - f(-1)}{1-(-1)} = 1$$

Εφαρμογή ότι ισχύει το ΘΜΤ

στο $(-1,1)$ οφρα οντν

$\exists \xi \in (-1,1)$ τ.ω $f'(\xi) = 1$

$\underline{x<0}$

$f'(\xi)=1$

$3x^2=1$

$x^2=\frac{1}{3}$

$x=\pm\frac{\sqrt{3}}{3}$

$x=-\frac{\sqrt{3}}{3}$

$\underline{x>0}$

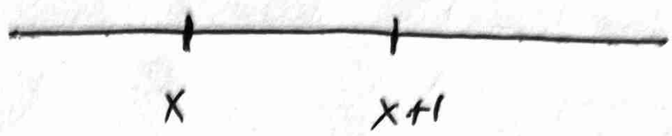
$f'(\xi)=1$

$2\xi=1$

$\xi=\frac{1}{2}$

5. 8 NSO $\exists \xi \in (x, x+1)$ T.W

$$f(x) + f'(\xi) = f(x+1).$$



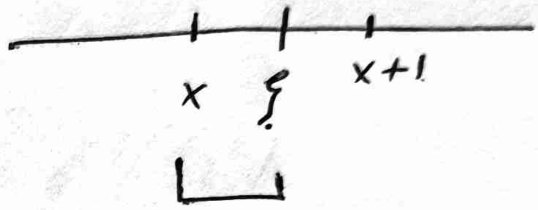
$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1 - x} = f(x+1) - f(x)$$

$$f'(\xi) + f(x) = f(x+1)$$

ΘΜΤ + Ανισότητα

A Έστω $f' \uparrow$

1. Νδο $f'(x) + f(x) < f(x+1)$



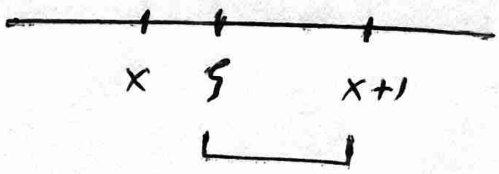
$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$$

$$x < \xi \xRightarrow{f' \uparrow} f'(x) < f'(\xi)$$

$$f'(x) < f(x+1) - f(x)$$

$$\boxed{f'(x) + f(x) < f(x+1)}$$

2. Νδο $f(x+1) - f'(x+1) < f(x)$



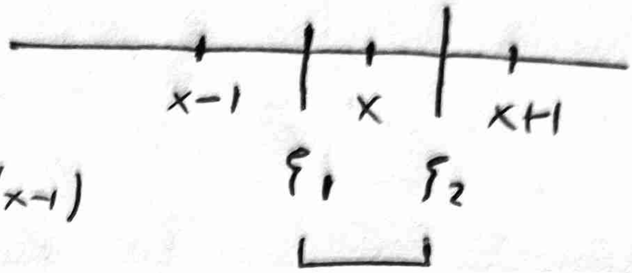
$$f'(\xi) = f(x+1) - f(x)$$

$$\xi < x+1 \xRightarrow{f' \uparrow} f'(\xi) < f'(x+1)$$

$$f(x+1) - f(x) < f'(x+1)$$

$$\boxed{f(x+1) - f'(x+1) < f(x)}.$$

3. Νδο $2f(x) < f(x+1) + f(x-1)$



$$f'(\xi_1) = \frac{f(x) - f(x-1)}{x - (x-1)} = f(x) - f(x-1)$$

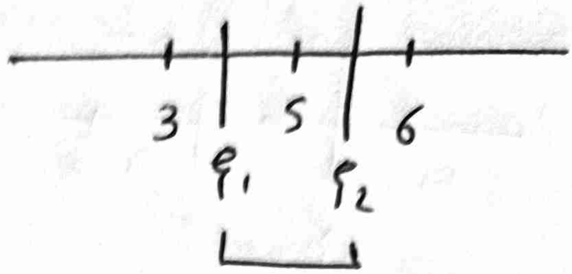
$$f'(\xi_2) = \frac{f(x+1) - f(x)}{x+1 - x} = f(x+1) - f(x)$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$f(x) - f(x-1) < f(x+1) - f(x)$$

$$\boxed{2f(x) < f(x+1) + f(x-1)}$$

4 Νδο $3f(5) < 2f(6) + f(3)$



$$f'(\xi_1) = \frac{f(5)-f(3)}{5-3} = \frac{f(5)-f(3)}{2}$$

$$f'(\xi_2) = \frac{f(6)-f(5)}{6-5} = f(6)-f(5)$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2) \Rightarrow \frac{f(5)-f(3)}{2} < f(6)-f(5)$$

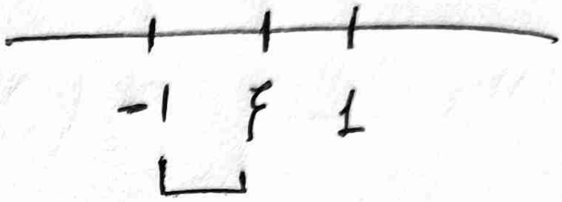
$$f(5)-f(3) < 2f(6) - 2f(5)$$

$$\boxed{3f(5) < 2f(6) + f(3)}$$

5 Αν η $y=-2x+3$ εφαπτεται της C_f στο $A(-1, f(-1))$

νδο $f(1)>1$.

$$\begin{cases} f(-1) = -2(-1)+3 = 5 \\ f'(-1) = -2 \end{cases}$$



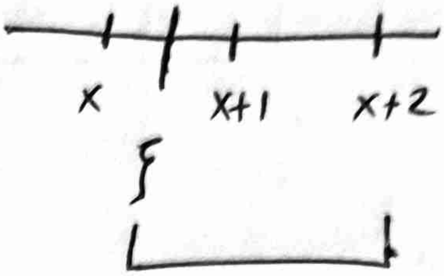
$$f'(\xi) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{f(1)-5}{2}$$

$$-1 < \xi \Rightarrow f'(-1) < f'(\xi) \Rightarrow -2 < \frac{f(1)-5}{2}$$

$$\Rightarrow -4 < f(1)-5$$

$$1 < f(1)$$

6. NTS $f(x+1) < f'(x+2) + f(x)$.



$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$$

$\xi < x+2$

$f' \nearrow$

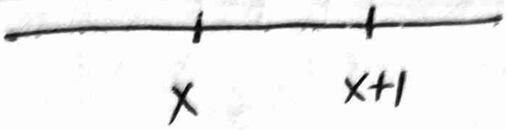
$f'(\xi) < f'(x+2)$

$f(x+1) - f(x) < f'(x+2)$

$f(x+1) < f'(x+2) + f(x)$

$\boxed{B}$ Εστω $|f'(x)| \leq 1$ Εστω $f' \nearrow$

Νδο $f(x+1) \leq f(x)$



$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$$

ομως γνωριζω οτι $|f'(x)| \leq 1 \ \forall x \in \mathbb{R}$.

$$-1 \leq f'(x) \leq 1$$

$$-1 \leq f'(\xi) \leq 1$$

$$-1 \leq f(x+1) - f(x) \leq 1$$

$$\underline{\underline{f(x+1) \leq f(x)}}$$

6. β Ν.δ.ο $\frac{1}{3} < \ln\frac{3}{2} < \frac{1}{2}$

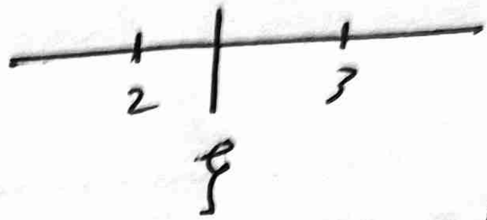
$$\frac{1}{3} < \ln 3 - \ln 2 < \frac{1}{2}$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} < 0$

$f' \searrow$



$$f'(\xi) = \frac{f(3) - f(2)}{3-2}$$

$$f'(\xi) = \frac{\ln 3 - \ln 2}{1}$$

$$f'(\xi) = \ln 3 - \ln 2$$

Ομως $\underline{\qquad}$

$$2 < \xi < 3$$

$$f'(2) > f'(\xi) > f'(3)$$

$$\frac{1}{2} > \ln 3 - \ln 2 > \frac{1}{3}$$

8) $$(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\frac{\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\beta} < (\beta-\alpha)\,\varepsilon\varphi\beta$$

$$0<\alpha<\beta<\frac{\pi}{2}$$

$$(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\sigma\upsilon\nu\alpha - \ln\sigma\upsilon\nu\beta < (\beta-\alpha)\,\varepsilon\varphi\beta$$

$$\boxed{f(x)=\ln\sigma\upsilon\nu x}$$

$$f'(x)=\frac{-\eta\mu x}{\sigma\upsilon\nu x}=-\varepsilon\varphi x$$

$$f''(x)=-\frac{1}{\sigma\upsilon\nu^2 x}<0$$

$f' \downarrow$

$$f'(\xi)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$$

$$\alpha<\xi<\beta$$

$$f'(\alpha)>f'(\xi)>f'(\beta)$$

$$-\varepsilon\varphi\alpha > \frac{f(\beta)-f(\alpha)}{\beta-\alpha} > -\varepsilon\varphi\beta.$$

$$-(\beta-\alpha)\,\varepsilon\varphi\alpha > \ln\sigma\upsilon\nu\beta - \ln\sigma\upsilon\nu\alpha > -(\beta-\alpha)\,\varepsilon\varphi\beta$$

$$(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\sigma\upsilon\nu\alpha - \ln\sigma\upsilon\nu\beta < (\beta-\alpha)\,\varepsilon\varphi\beta$$

7. $\forall x>0$ νδο $\dfrac{1}{x+1} < \ln\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}$

<u>α' τρόπος</u>

$\ln x \leq x-1 \Rightarrow \ln\left(1+\frac{1}{x}\right) < 1+\frac{1}{x} - 1$

$$\boxed{\ln\left(1+\frac{1}{x}\right) < \frac{1}{x}}$$

Άρκει νδο $\dfrac{1}{x+1} < \ln\left(1+\dfrac{1}{x}\right)$ $\forall x>0$

$$\underbrace{\frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right)}_{\varphi(x)} < 0$$

$$\varphi'(x) = -\frac{1}{(x+1)^2} - \frac{-\frac{1}{x^2}}{1+\frac{1}{x}} = -\frac{1}{(x+1)^2} + \frac{1}{x^2+x}$$

$$\varphi'(x) = \frac{(x+1)^2 - x^2 - x}{(x+1)^2(x^2+x)} = \frac{\cancel{x^2}+2x+1-\cancel{x^2}-x}{(x+1)^2(x^2+x)} = \frac{x+1}{(x+1)^2(x^2+x)}$$

$\varphi'(x) > 0 \Rightarrow \varphi \nearrow$

$$\lim_{x \to 0^+} \varphi(x) = \lim_{x \to 0^+} \frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right) = 1 - \infty$$

$$= -\infty$$

$$\lim_{x \to +\infty} \varphi(x) = \lim_{x \to +\infty} \frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right) = 0$$

$$\Sigma T \varphi = (-\infty, 0)$$

$$\Rightarrow \varphi(x) < 0$$

$$\frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right) < 0$$

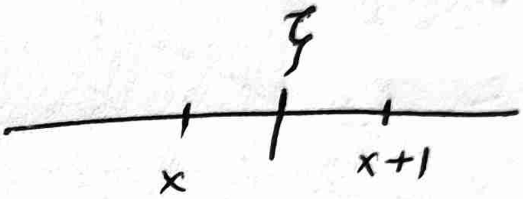
$$\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right)$$

Β' τρόπος

$$\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$$

$$\frac{1}{x+1} < \ln\left(\frac{x+1}{x}\right) < \frac{1}{x}$$

$$\boxed{\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}}$$



$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f' \downarrow$

$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x}$$

$$x < \xi < x+1$$

$$f' \downarrow$$

$$f'(x) > f'(\xi) > f'(x+1)$$

$$\frac{1}{x} > f(x+1) - f(x) > \frac{1}{x+1}$$

$$\frac{1}{x} > \ln(x+1) - \ln x > \frac{1}{x+1}$$

19. $f''(x) \neq 0 \quad \forall x \in R$.

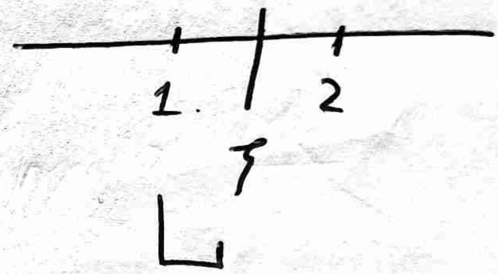
$f(2) - f(1) > f'(1)$

Νδο $f''(x) > 0$.

f συν + παρ
f' συν + παρ
f'' συν.

Αφού f'' συνεχής και $f''(x) \neq 0$

$f''(x) > 0$ ή $f''(x) < 0 \quad \forall x \in R$



$$f'(\xi) = \frac{f(2) - f(1)}{2 - 1}$$

$$f'(\xi) = f(2) - f(1)$$

γνωρίζω ότι $f(2) - f(1) > f'(1)$

$f'(\xi) > f'(1)$

$$f''(\rho) = \frac{\overset{(+)}{f'(\xi) - f'(1)}}{\underset{(+)}{\xi - 1}} > 0$$

$\underline{\underline{f''(x) > 0}}$