# Ενότητα 10

**11.** Δίνεται $f(x)=\sqrt{x^2+1}$

ⓐ $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\sqrt{x^2+1} = +\infty$

ⓑ $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\sqrt{x^2+1} = +\infty$

ⓒ $\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to+\infty}\frac{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x}$

$= \lim_{x\to+\infty}\frac{\cancel{x}\sqrt{1+\frac{1}{x^2}}}{\cancel{x}} = 1.$

ⓓ $\lim_{x\to+\infty}(f(x)-x) = \lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right) =$

$= \lim_{x\to+\infty}\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty}\frac{\sqrt{x^2+1}^{\,2}-x^2}{\sqrt{x^2+1}+x}$

$= \lim_{x\to+\infty}\frac{1}{\sqrt{x^2+1}+x} = 0.$

(ε) $$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1+\frac{1}{x}\right)}}{x}$$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{1+\frac{1}{x}}}{\cancel{x}} = \underline{\underline{-1}}$$

(στ) $$\lim_{x\to-\infty} f(x)+x = \lim_{x\to-\infty} \sqrt{x^2+1}+x =$$

$$= \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+1}+x\right)\left(\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}-x} =$$

$$= \lim_{x\to-\infty} \frac{\cancel{\sqrt{x^2+1}}^{\cancel{2}} - x^2}{\sqrt{x^2+1}-x} = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}-x}$$

$$= 0$$

**28.** (a) $\lim\limits_{x\to+\infty}\left(ημ\frac{1}{x} - συν\frac{2}{x}\right) = ημ0 - συν0 = 0-1 = -1$

**29.** (a) $\lim\limits_{x\to+\infty}\frac{xημx}{x^2+1} = 0$ .

$-1 \le ημx \le 1$

Το $x>0$ κοντά στο $+\infty$

$-x \le xημx \le x$

$$\boxed{-\frac{x}{x^2+1} \le \frac{xημx}{x^2+1} \le \frac{x}{x^2+1}}$$

$$\lim_{x\to+\infty} -\frac{x}{x^2+1} = \lim_{x\to+\infty} -\frac{\cancel{x}}{x^{\cancel{2}}} = 0$$

$$\lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{\cancel{x}}{x^{\cancel{2}}} = 0$$

Άρα κ.π. $\lim\limits_{x\to+\infty}\frac{xημx}{x^2+1} = 0$

**30.** (α) $\lim_{x \to -\infty} x\,\eta\mu\frac{1}{x} = \lim_{x\to-\infty} \cancel{x}\,\dfrac{\eta\mu\frac{1}{x}}{\frac{1}{x}}\,\dfrac{1}{\cancel{x}} = 1$

(β) $\lim_{x\to-\infty} \dfrac{\eta\mu x}{x} = 0$.

$-1 \le \eta\mu x \le 1$

$$\boxed{-\frac{1}{x} \geq \frac{\eta\mu x}{x} \geq \frac{1}{x}}$$

bούσα στο $-\infty$

το $x<0$

$$\left.\begin{aligned} &\lim_{x\to-\infty} -\frac{1}{x} = 0 \\ &\lim_{x\to-\infty} \frac{1}{x} = 0 \end{aligned}\right\} \text{Από K.Π}$$

$\lim_{x\to-\infty} \dfrac{\eta\mu x}{x} = 0$.

**31.** (α) $\lim_{x\to+\infty} \dfrac{x+\eta\mu x}{x+1} = 1$ Από K.Π.

$-1 \le \eta\mu x \le 1$

$x-1 \le x+\eta\mu x \le x+1$

$$\boxed{\frac{x-1}{x+1} \le \frac{x+\eta\mu x}{x+1} \le 1}$$

• $\lim_{x\to+\infty} \dfrac{x-1}{x+1} = 1$.

$\lim_{x\to+\infty} 1 = 1$

**32.** (a) $\lim\limits_{x \to +\infty} x^2 + \eta\mu x = +\infty$

$-1 \leq \eta\mu x \leq 1$

$$\boxed{x^2 - 1 \leq x^2 + \eta\mu x \leq x^2 + 1}$$

$\left.\begin{array}{l} \lim\limits_{x \to +\infty} x^2 - 1 = +\infty \\ \lim\limits_{x \to +\infty} x^2 + 1 = +\infty \end{array}\right\}$ Απο Κ.Π $\lim\limits_{x \to +\infty} x^2 + \eta\mu x = +\infty$

**33.** (a) $\lim\limits_{x \to +\infty} \dfrac{x}{2 + \eta\mu x}$

$-1 \leq \eta\mu x \leq 1$

$1 \leq 2 + \eta\mu x \leq 3$

$\dfrac{1}{2} \geq \dfrac{1}{2 + \eta\mu x} \geq \dfrac{1}{3}$

$x > 0$ στο $+\infty$

$$\boxed{\frac{x}{2} \geq \frac{x}{2 + \eta\mu x} \geq \frac{x}{3}}$$

$\lim\limits_{x \to +\infty} \dfrac{x}{2} = +\infty$ $\lim\limits_{x \to +\infty} \dfrac{x}{3} = +\infty$,

34. ⓐ $\lim_{x \to +\infty} \left(\frac{e}{3}\right)^x = 0$

ⓑ $\lim_{x \to -\infty} \left(\frac{1}{3}\right)^x = 0$

35. ⓐ $\lim_{x \to +\infty} (e^x - 2^x + 1) = \lim_{x \to +\infty} e^x\left(1 - \frac{2^x}{e^x} + \frac{1}{e^x}\right)$

$= \lim_{x \to +\infty} e^x\left(1 - \cancel{\left(\frac{2}{e}\right)^x} + \cancel{\frac{1}{e^x}}\right) = +\infty(1 - 0 + 0) = +\infty$

ⓑ $\lim_{x \to +\infty} (e^x - 3^{x+1} - 2) = \lim_{x \to +\infty} (e^x - 3^x \cdot 3 - 2)$

$= \lim_{x \to +\infty} 3^x\left(\frac{e^x}{3^x} - 3 - \frac{2}{3^x}\right) = \lim_{x \to +\infty} 3^x\left(\cancel{\left(\frac{e}{3}\right)^x} - 3 - \frac{2}{3^x}\right)$

$= +\infty(0 - 3 - 0)$

$= -\infty$.

**36.** ⓐ $\lim\limits_{x\to+\infty} \dfrac{3^x+3\cdot 2^x}{3^x-2^x} = \lim\limits_{x\to+\infty} \dfrac{\cancel{3^x}\left(1+3\cdot\left(\frac{2}{3}\right)^x\right)}{\cancel{3^x}\left(1-\left(\frac{2}{3}\right)^x\right)}$

$= \dfrac{1+3\cdot 0}{1-0} = 1.$

ⓑ $\lim\limits_{x\to-\infty} \dfrac{3^{x+1}+5e^x}{2\cdot 3^x-e^x} = \lim\limits_{x\to-\infty} \dfrac{3^x\cdot 3+5e^x}{2\cdot 3^x-e^x}$

$= \lim\limits_{x\to-\infty} \dfrac{\cancel{e^x}\left(\left(\frac{3}{e}\right)^x\cdot 3+5\right)}{\cancel{e^x}\left(2\cdot\left(\frac{3}{e}\right)^x-1\right)} = \dfrac{0+5}{0-1} = -5.$

**38.** ⓐ $\lim\limits_{x\to-\infty} \dfrac{e^x}{x} = \lim\limits_{x\to-\infty} e^x\cdot\dfrac{1}{x} = 0\cdot 0 = 0$

ⓑ $\lim\limits_{x\to-\infty} \dfrac{x}{e^x} = \lim\limits_{x\to-\infty} x\cdot\dfrac{1}{e^x} = -\infty\cdot(+\infty) = -\infty,$

39. (α) $\lim_{x\to+\infty} e^{-x} = e^{-\infty} = 0$.

(β) $\lim_{x\to-\infty} e^{-x} = e^{+\infty} = +\infty$.

(δ) $\lim_{x\to-\infty} e^{-x^2} = e^{-\infty} = 0$

(στ) $\lim_{x\to0^-} e^{\frac{1}{x}} = e^{-\infty} = 0$.

40. (α) $\lim_{x\to+\infty} \left(e^{x} + \frac{1}{x} - 1\right) = +\infty + 0 - 1 = +\infty$.

(γ) $\lim_{x\to+\infty} \left(e^{-x} + \frac{1}{x+1} - 1\right) = e^{-\infty} + 0 - 1 = 0 - 1 = -1$

(ε) $\lim_{x\to0} \frac{1}{e^{|x|} - 1} = +\infty$

• $|x| \geq 0 \Rightarrow e^{|x|} \geq e^{0} \Rightarrow e^{|x|} \geq 1$

$e^{|x|} - 1 \geq 0$

**41.** (a) $\lim\limits_{x\to-\infty} e^x ημx$

$-1 \le ημx \le 1$

$$\boxed{-e^x \le e^x ημx \le e^x}$$

$\left.\begin{array}{l} \lim\limits_{x\to-\infty} -e^x = 0 \\ \\ \lim\limits_{x\to-\infty} e^x = 0 \end{array}\right\}$ Από Κ.Π $\lim\limits_{x\to-\infty} e^x ημx = 0$.

**42.** (a) $\lim\limits_{x\to+\infty} e^x + ημx$

$-1 \le ημx \le 1$

$$\boxed{e^x - 1 \le e^x + ημx \le e^x + 1}$$

$\left.\begin{array}{l} \lim\limits_{x\to+\infty} e^x - 1 = +\infty \\ \\ \lim\limits_{x\to+\infty} e^x + 1 = +\infty \end{array}\right\}$ Από Κ.Π $\lim\limits_{x\to+\infty} e^x + ημx = +\infty$.

**43.** (α) $\lim\limits_{x \to 0} x^2+1+\ln x = 0+1+(-\infty) = -\infty$

(γ) $\lim\limits_{x \to 0} \ln x - \dfrac{1}{e^x-1} = \lim\limits_{x \to 0^+} \ln x - \dfrac{1}{e^x-1}$

$= -\infty - (+\infty) = -\infty - \infty = -\infty$

Αφού $x>0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1$

$e^x - 1 > 0$

**44.** (α) $\lim\limits_{x \to 0} \dfrac{3x+1}{\ln x} = \dfrac{1}{-\infty} = 0$

(β) $\lim\limits_{x \to 0} \dfrac{2x-5}{\ln x} = \dfrac{-5}{-\infty} = 0$

**45.** (α) $\lim\limits_{x\to 0} \dfrac{1+\ln x}{x} = \lim\limits_{x\to 0^+} (1+\ln x)\, \dfrac{1}{x} =$

$= (1-\infty)\cdot(+\infty) = (-\infty)(+\infty) = -\infty$ ,

(β) $\lim\limits_{x\to 0} \dfrac{\sigma\upsilon\nu x-1}{\ln x} = \lim\limits_{x\to 0} (\sigma\upsilon\nu x-1)\, \dfrac{1}{\ln x} = 0\cdot\dfrac{1}{\infty} =$

$= 0\cdot 0 = 0$ .

**46.** (α) $\lim\limits_{x\to +\infty} \left(\ln x - \dfrac{1}{x} - 1\right) = +\infty - 0 - 1 = +\infty$ .

(γ) $\lim\limits_{x\to +\infty} x\ln x = +\infty$ .

**47.** (α) $\lim\limits_{x\to +\infty} \ln x + \eta\mu x = +\infty$

$-1 \le \eta\mu x \le 1$

$\boxed{\ln x - 1 \le \ln x + \eta\mu x \le \ln x + 1}$

$\lim\limits_{x\to +\infty} \ln x - 1 = +\infty$ $\qquad$ $\lim\limits_{x\to +\infty} \ln x + 1 = +\infty$ .

Από κ.π. $\lim\limits_{x\to +\infty} \ln x + \eta\mu x = +\infty$

48. ⓐ $\lim_{x \to 1^-} \ln \frac{x}{1-x} = \ln(+\infty) = +\infty$

$\rightarrow \lim_{x \to 1^-} \frac{x}{1-x} = \lim_{x \to 1^-} x \cdot \frac{1}{1-x} = 1 \cdot (+\infty) = +\infty$

ⓓ. $\lim_{x \to -\infty} \ln(1-e^x) = \ln(1-e^{-\infty}) =$

$= \ln(1-0) = \ln 1 = 0$

ⓔ. $\lim_{x \to -\infty} \ln(1+x^2) - \ln(2-x) =$

$= \lim_{x \to -\infty} \ln \frac{1+x^2}{2-x} = \ln +\infty = +\infty$

$\rightarrow \lim_{x \to -\infty} \frac{1+x^2}{2-x} = \lim_{x \to -\infty} \frac{x^{\cancel{2}}}{-\cancel{x}} = \lim_{x \to -\infty} -x = +\infty$

# Ενότητα 12

**26.** $f:[0,1] \longrightarrow \mathbb{R}$ συνεχής

$4 < f(x) < 5 \quad \forall x \in [0,1)$

Νδο η εξίσωση $f^2(x) - 5f(x) + 4x = 0$

έχει ρίζα $(0,1)$.

$$\boxed{g(x) = f^2(x) - 5f(x) + 4x}$$

Η $g(x)$ συνεχής στο $[0,1]$ ως π.σ.σ.

$g(0) = f^2(0) - 5f(0) + 4 \cdot 0$ $\qquad 4 < f(0) < 5$

$\qquad -1 < f(0) - 5 < 0$

$g(0) = f(0)\,(f(0) - 5) < 0$

$g(1) = f^2(1) - 5f(1) + 4 = (f(1) - 4)(f(1) - 1) > 0$

$g(0)g(1) < 0$ από Bolzano $\qquad 4 < f(1) < 5$

$\qquad 0 < f(1) - 4 < 1$

$\exists\, \xi \in (0,1)$ τ.ω $g(\xi) = 0$ $\qquad 4 < f(1) < 5$

$f^2(\xi) - 5f(\xi) + 4\xi = 0$ $\qquad 3 < f(1) - 1 < 4$

**27.** Νδο η εξισωση $x^3-(\kappa\lambda-2)x+1=0$

εχει ριζα $(-1,0)$ $\kappa+\lambda=2$,

$\underline{\lambda=2-\kappa}$

$\boxed{f(x)=x^3-(\kappa\lambda-2)x+1}$

Η $f(x)$ συνεχη $[-1,0]$ ως π.σ.σ

$f(-1)=-1+\kappa\lambda-2+1=\kappa\lambda-2=$

$f(0)=1>0$

$=\kappa(2-\kappa)-2=$

$=2\kappa-\kappa^2-2=0$

Αρα $f(-1)f(0)<0$

$\Leftrightarrow$

$=-\kappa^2+2\kappa-2<0$

$\Delta=-4<0$

Bolzano

$\exists\,\xi\in(-1,0)$

τ.ω $f(\xi)=0$

$\xi^3-(\kappa\lambda-2)\xi+1=0$ ✓

28. $f$ συνεχής $[\alpha, \beta]$

$f(\alpha) \neq 0$

Ν.δ.ο. $\exists x_0 \in (\alpha, \beta)$ τ.ω. $\dfrac{f(x_0)}{x_0-\alpha} = \dfrac{f(\alpha)+f(\beta)}{\beta-\alpha}$.

$$\frac{f(x)}{x-\alpha} = \frac{f(\alpha)+f(\beta)}{\beta-\alpha}$$

$$(\beta-\alpha) f(x) = (x-\alpha)\big(f(\alpha)+f(\beta)\big)$$

$$\underbrace{(\beta-\alpha) f(x) - (x-\alpha)\big(f(\alpha)+f(\beta)\big)}_{\varphi(x)} = 0$$

Η $\varphi(x)$ συνεχής στο $[\alpha, \beta]$ ως π.σ.σ.

$\varphi(\alpha) = (\beta-\alpha) f(\alpha)$

$\varphi(\beta) = (\beta-\alpha) f(\beta) - (\beta-\alpha)\big(f(\alpha)+f(\beta)\big) =$

$= (\beta-\alpha)\big(\cancel{f(\beta)} - f(\alpha) - \cancel{f(\beta)}\big) = -f(\alpha)(\beta-\alpha)$.

$$\varphi(a)\,\varphi(B) = -(B-a)^2 f^2(a) < 0$$

Bolzano $\exists\, x_0 \in (a, B)$

T.W $\varphi(x_0) = 0$

$$\frac{f(x_0)}{x_0 - a} = \frac{f(a) + f(B)}{B - a}$$

**29.** Νδο η εξίσωση $x\,εφx = 1$ έχει 2 τουλ. ρίζες $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

| $x$ | $-\frac{\pi}{2}$ | $0$ | $\frac{\pi}{2}$. |
|---|---|---|---|
| | $+$ | $-$ | $+$ |

$$x\,\frac{ημx}{συνx} = 1$$

$$x\,ημx = συνx$$

$$\underbrace{x\,ημx - συνx}_{f(x)} = 0$$

Η $f(x)$ συνεχής $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ως π.σ.σ.

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}\,ημ\left(-\frac{\pi}{2}\right) - συν\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$f(0) = -1$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$f(-\frac{\pi}{2}) \cdot f(0) < 0$ Bolzano $\exists \xi_1 \in (-\frac{\pi}{2}, 0)$

τ.ω $f(\xi_1) = 0$

$f(0) \cdot f(\frac{\pi}{2}) < 0$ Bolzano $\exists \xi_2 \in (0, \frac{\pi}{2})$

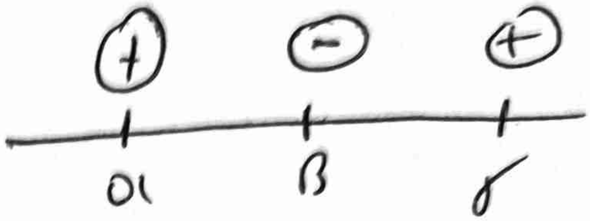
τ.ω $f(\xi_2) = 0$,

# 30. Νδο η εξίσωση

$$\underbrace{(x-\beta)(x-\gamma)+2(x-\alpha)(x-\gamma)}_{f(x)}+3(x-\alpha)(x-\beta)=0$$

$\alpha<\beta<\gamma$.

Η $f(x)$ συνεχής στο $[\alpha,\gamma]$ ως π.δ.σ



$f(\alpha)=(\alpha-\beta)(\alpha-\gamma)>0$

$f(\beta)=2(\beta-\alpha)(\beta-\gamma)<0$

$f(\gamma)=3(\gamma-\alpha)(\gamma-\beta)>0$.

$f(\alpha)f(\beta)<0$ Bolzano $\exists\, \xi_1\in(\alpha,\beta)$ τ.ω $f(\xi_1)=0$.

$f(\beta)f(\gamma)<0$ Bolzano $\exists\, \xi_2\in(\beta,\gamma)$ τ.ω $f(\xi_2)=0$.

Η $f(x)$ είναι πολυώνυμο 2ου βαθμού άρα έχει το πολύ 2 λύσεις και αφού βρήκα τουλάχιστον 2 έχει 2.

**31.** $f(x) = (x-2)e^x - (x+2)$

(α) ν.δ.ο η $f$ έχει ρίζα στο $(1,3)$

$f$ συνεχής $[1,3]$ ως π.σ.ν

$f(1) = -e - 3 < 0$

$f(3) = e^3 - 5 > 0$

$\left.\right\}$ $f(1)f(3) < 0$

Bolzano

$\exists\ \xi$ τ.ω $f(\xi) = 0$,

$\xi \in (1,3)$

(β) Αρκεί ν.δ.ο $f(-\xi) = 0$.

$$f(-\xi) = (-\xi-2)e^{-\xi} - (-\xi+2) =$$

$$= -\frac{(\xi+2)}{e^{\xi}} + (\xi-2) = -\frac{(\xi+2) + (\xi-2)e^{\xi}}{e^{\xi}}$$

$$= -\frac{f(\xi)}{e^{\xi}} = 0$$

Άρα $f(\xi)=0$ $\quad \xi\in(1,3)$

$f(-\xi)=0$ $\quad -\xi\in(-3,-1)$

Έχει δυο ρίζες αντίθετες

**32.** $f: \mathbb{R} \to \mathbb{R}$ συνεχής

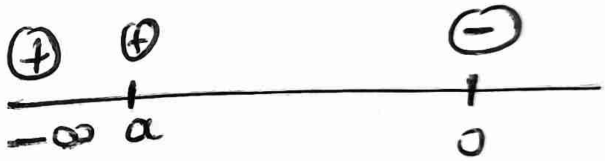
$f(0) < 1$

$\lim_{x \to -\infty} f(x) = +\infty$

Ν.δ.ο $\exists x_0 < 0$ τ.ω $f(x_0) = e^{x_0} + x_0 \,\eta\mu \frac{1}{x_0}$

$$f(x) = e^{x} + x\,\eta\mu\frac{1}{x}$$

$$\underbrace{f(x) - e^{x} - x\,\eta\mu\frac{1}{x}}_{\varphi(x)} = 0$$




$$\lim_{x \to -\infty}\left(f(x) - e^{x} - x\,\eta\mu\frac{1}{x}\right) = +\infty - e^{-\infty} - 1 = +\infty - 1 = +\infty$$

$$\rightarrow \lim_{x \to -\infty} x\,\eta\mu\frac{1}{x} = \lim_{x \to -\infty} \frac{\cancel{x}\,\eta\mu\frac{1}{x}}{\frac{1}{x}}\,\cancel{\frac{1}{x}} = 1$$

$\exists\, \alpha$ κοντά στο $-\infty$ τ.ω $\varphi(\alpha) > 0$

$$\lim_{x\to 0^-} \varphi(x) = \lim_{x\to 0^-} f(x) - e^x - x\,\eta\mu\frac{1}{x} =$$

$$= f(0) - 1 - 0 = f(0) - 1 < 0$$

γιατί

$f(0) < 1$

$$\to \lim_{x\to 0^-} x\,\eta\mu\frac{1}{x} = 0$$

$$-1 \leq \eta\mu\frac{1}{x} \leq 1$$

$$\boxed{-x \geq x\,\eta\mu\frac{1}{x} \geq x}$$

$$\left.\begin{aligned} &\lim_{x\to 0^-} -x = 0 \\ &\lim_{x\to 0^-} x = 0 \end{aligned}\right\} \lim_{x\to 0^-} x\,\eta\mu\frac{1}{x} = 0$$

$\exists$ $6 < 0$ κοντά στο $0^-$ τ.ω

$$\varphi(6) < 0$$

$\varphi(a)\varphi(b)<0$ Bolzano $\exists x_0 \in (a,b)$

τ.ω $\varphi(x_0)=0$

$$f(x_0) = e^{x_0} + x_0\, \eta\mu \frac{1}{x_0}.$$

**34.** $f: \mathbb{R} \to \mathbb{R}$ συνεχής γν. μονότονη.

$f(0) = 0$

$f(1) = 1$

Νδο $\exists x_0 \in (0,1)$ μοναδικό τ.ω

$$f(x_0) = 1 - x_0^3$$

$$f(x) = 1 - x^3$$

$$\underbrace{f(x) - 1 + x^3}_{\varphi(x)} = 0$$

$$\varphi(0) = f(0) - 1 = 0 - 1 = -1 < 0$$

$$\varphi(1) = f(1) - 1 + 1 = 1 > 0$$

$\varphi(0)\varphi(1) < 0$ Bolzano $\exists x_0 \in (0,1)$

τ.ω $\varphi(x_0) = 0 \Rightarrow f(x_0) = 1 - x_0^3$.

$$\left.\begin{aligned} &\bullet\ x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \\ &\bullet\ x_1 < x_2 \rightarrow x_1^3 - 1 < x_2^3 < 1 \end{aligned}\right\} \oplus$$

$\varphi \nearrow$

άρα το $x_0$ μοναδικό.

Αφού $f$ γν. μονότονη $\Rightarrow f \nearrow$ ή $f \searrow$

$$\left.\begin{aligned} &0 < 1 \\ &\underset{\substack{\| \\ 0}}{f(0)} < \underset{\substack{\| \\ 1}}{f(1)} \end{aligned}\right\} f \nearrow$$

# 35. $f(x) = 2x^3 + x - 1$

(α) Νδο η $2x^3 = 1 - x$ έχει μον. ρίζα $(0,1)$.

$$2x^3 + x - 1 = 0$$

$$f(x) = 0$$

Η $f$ συνεχής $[0,1]$ ως π.σ.σ.

$$\left.\begin{aligned} f(0) &= -1 \\ f(1) &= 2 \end{aligned}\right\} f(0)f(1) < 0 \quad \text{Bolzano}$$

$\exists\, x_0 \in (0,1)$ τ.ω

$f(x_0) = 0$.

Επειδή $f \nearrow$ το $\exists$ μοναδικό.

(β) $$\lim_{x \to x_0} \frac{1}{f(x)} \quad \begin{cases} \lim\limits_{x \to x_0^-} \dfrac{1}{f(x)} = -\infty \\ \lim\limits_{x \to x_0^+} \dfrac{1}{f(x)} = +\infty \end{cases}$$

| $x$ | $x_0$ |
|---|---|
| $f(x)$ | $\nearrow$ $-$ $0$ $+$ $\nearrow$ |

Το όριο δεν υπάρχει!

36. Νδο η $e^x + x = a + \sigma\upsilon\nu x$, $a>0$

έχει τουλάχιστον μια ρίζα $(0,a)$.

$$\boxed{f(x) = e^x + x - a - \sigma\upsilon\nu x}$$

Η $f(x)$ συνεχής $[0,a]$ ως π.σ.σ.

$f(0) = 1 + 0 - a - 1 = -a < 0$

$f(a) = e^a + a - a - \sigma\upsilon\nu a = e^a - \sigma\upsilon\nu a =$

$$= \underbrace{e^a - 1}_{+} + \underbrace{1 - \sigma\upsilon\nu a}_{+} > 0.$$

$a > 0$

$e^a > e^0$

$e^a > 1$

$e^a - 1 > 0$

$-1 \leq \sigma\upsilon\nu a \leq 1$

$1 \geq -\sigma\upsilon\nu a \geq -1$

$2 \geq 1 - \sigma\upsilon\nu a \geq 0$

$f(0)f(a) < 0$ Bolzano $\exists\, \xi \in (0,a)$ τ.ω. $f(\xi) = 0$.

**37.** $f: \mathbb{R} \to \mathbb{R}$ συνεχής.

$$x f(x) + 2 = f(x) + \sqrt{3x^2+1} \quad \forall x \in \mathbb{R}. \quad (*)$$

Ν.δ.ο $\exists x_0 \in (0,1)$ τ.ω $4f(x_0) = 7x_0$

$$4f(x) = 7x$$

$$\underbrace{4f(x) - 7x}_{\varphi(x)} = 0$$

Η $\varphi(x)$ συνεχής $[0,1]$ ως π.σ.σ.

$$\varphi(0) = 4f(0) = 4 > 0$$

$$\varphi(1) = 4f(1) - 7 = 4 \cdot \frac{3}{2} - 7 = -1 < 0$$

$(*)$ $x = 0$

$$2 = f(0) + 1$$

$$f(0) = +1$$

$$x f(x) - f(x) = \sqrt{3x^2+1} - 2$$

$$(x-1) f(x) = \sqrt{3x^2+1} - 2$$

Για $x \neq 1$ $\quad f(x) = \dfrac{\sqrt{3x^2+1} - 2}{x-1}$

$$f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sqrt{3x^2+1}-2}{x-1}$$

f συνεχής

$$= \lim_{x\to 1} \frac{\sqrt{3x^2+1}^{\,2} - 2^2}{(x-1)\left(\sqrt{3x^2+1}+2\right)} = \lim_{x\to 1} \frac{3\cancel{(x-1)}(x+1)}{\cancel{(x-1)}\left(\sqrt{3x^2+1}+2\right)}$$

$$= \frac{6}{4} = \frac{3}{2}.$$

Άρα $\varphi(0)\,\varphi(1) < 0$ Bolzano

$\exists\, x_0 \in (0,1)$ τ.ω $\varphi(x_0) = 0$

$4f(x_0) = 7x_0,$

**42.** $f: [a, 2a) \longrightarrow \mathbb{R}$, $a>0$ συνεχ.

$f(a) \neq 0$ $\qquad$ $Bf(a) + e^{B} f(2a) = f(2a)$

$B \neq 0$

Νδο η εξίσωση $f(x) = 0$ έχει ρίζα $(a, 2a)$.

Η $f$ συνεχ $[a, 2a]$ ωs Θ.Β.Γ

$f(a) =$

$f(2a) =$

$$B f(a) = f(2a) - e^{B} f(2a)$$

$$B f(a) = f(2a)\left(1 - e^{B}\right)$$

$$f(a) = f(2a) \frac{1 - e^{B}}{B}.$$

Γνωρίζω ότι $B \neq 0 \Rightarrow B > 0$ ή $B < 0$

Επίσης $f(a) \neq 0 \Rightarrow f(a) > 0$ ή $f(a) < 0$.

$$\boxed{f(a) = f(2a)\,\frac{1-e^B}{B} \quad \text{ισχύει}}$$

→ Αν $B > 0 \Rightarrow e^B > e^0 \Rightarrow e^B > 1 \Rightarrow 1 - e^B < 0$

άρα $\frac{1-e^B}{B} < 0$ συνεπώς αν $f(2a) > 0$

τότε $f(a) < 0$

$\Rightarrow f(a) f(2a) < 0$

Bolzano $\exists\, \xi \in (a, 2a)$ τ.ω $f(\xi) = 0$.

→ Αν $B < 0 \Rightarrow 1 - e^B > 0 \Rightarrow \frac{1-e^B}{B} < 0$

άρα αν $f(2a) > 0$ το $f(a) < 0$

ενώ αν $f(2a) < 0$ το $f(a) > 0$

$\Rightarrow f(a) f(2a) < 0$ ✓

39. $f: \mathbb{R} \to \mathbb{R}$ ciągła $f \downarrow$

$a > 0$

NDO $\exists x_0 \in (a, 3a)$ ponadto t.ż.

$$f(a) + f(3a) = 2f(x_0)$$

$$\underbrace{f(a) + f(3a) - 2f(x)}_{\varphi(x)} = 0$$

$$\varphi(a) = f(a) + f(3a) - 2f(a) = \underset{-}{f(3a) - f(a)}$$

$3a > a \Rightarrow f(3a) < f(a) \Rightarrow f(3a) - f(a) < 0$

$$\varphi(3a) = f(a) + f(3a) - 2f(3a) = \underset{+}{f(a) - f(3a)}$$

$\varphi(a)\varphi(3a) < 0$ Bolzano $\exists \xi \in (a, 3a)$

t.ż. $\varphi(\xi) = 0$.

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \Rightarrow -2f(x_1) < -2f(x_2)$

$$f(a) + f(3a) - 2f(x_1) < f(a) + f(3a) - 2f(x_2)$$

$$\varphi(x_1) < \varphi(x_2)$$

# Επομενο Μαθημα

Δευτερα 9 - 11:30

## 11

4 α δ

5 α β γ

7 ολα

9

10 .

11

12

13

20 α γ δ

21 α β γ

23.