Ενότητα 22

2. $f: (0,+\infty) \longrightarrow \mathbb{R}$ παρ/μη

$$f'(x) = \frac{1 - xf(x)}{x^2}, \quad \forall x > 0.$$

α) Ν.δ.ο. η $g(x) = xf(x) - \ln x$ σταθερή

$$g'(x) = f(x) + x f'(x) - \frac{1}{x}$$

$$g'(x) = f(x) + x \cdot \frac{1 - xf(x)}{x^2} - \frac{1}{x}$$

$$g'(x) = f(x) + \frac{1 - xf(x)}{x} - \frac{1}{x}$$

$$g'(x) = f(x) + \cancel{\frac{1}{x}} - \frac{\cancel{x}f(x)}{\cancel{x}} - \cancel{\frac{1}{x}}$$

$$g'(x) = f(x) - f(x) = 0 \Rightarrow g'(x) = \underline{g(x) = c}$$

$f(1) = 0$

β) $g(x) = c$

$xf(x) - \ln x = c$

$\xRightarrow{x=1}$ $f(1) - \ln 1 = c \Rightarrow 0 - 0 = c \Rightarrow c = 0$

$xf(x) - \ln x = 0$

$$f(x) = \frac{\ln x}{x}$$

3.

$f: \mathbb{R} \to \mathbb{R}$ παρ/μη.

$f(x) \neq 0$

$f'(x) = -2x f^2(x)$

α) Νδο $g(x) = \dfrac{1}{f(x)} - x^2$ σταθερή

$$g'(x) = \frac{-f'(x)}{f^2(x)} - 2x$$

$$g'(x) = -\frac{-2x\cancel{f^2(x)}}{\cancel{f^2(x)}} - 2x$$

$$g'(x) = 2x - 2x \Rightarrow \underline{g'(x) = 0}$$

$$\underline{\underline{g(x) = c}}$$

β) Αν $f(0) = 1$

Αφού $g(x) = c$

$$\frac{1}{f(x)} - x^2 = c$$

$\underline{x = 0}$

$$\frac{1}{f(0)} - 0^2 = c$$

$$1 = c$$

$$\frac{1}{f(x)} - x^2 = 1$$

$$\frac{1}{f(x)} = x^2 + 1$$

$$f(x) = \frac{1}{x^2+1}$$

5. $f : [0,\pi] \to \mathbb{R}$ $\quad f(0) = 0 \quad$ συνεχ

$$f'(x) = x\,\eta\mu x \qquad x \in (0,\pi)$$

Νδο $\quad f(x) = \eta\mu x - x\,\sigma\upsilon\nu x$

$$\underbrace{f(x) - \eta\mu x + x\,\sigma\upsilon\nu x}_{g(x)} = 0$$

$$g'(x) = f'(x) - \cancel{\sigma\upsilon\nu x} + \cancel{\sigma\upsilon\nu x} - x\,\eta\mu x$$

$$g'(x) = f'(x) - x\,\eta\mu x$$

$$g'(x) = x\,\eta\mu x - x\,\eta\mu x$$

$$g'(x) = 0$$

$$g(x) = c$$

$$f(x) - \eta\mu x + x\,\sigma\upsilon\nu x = c$$

$$\underline{x = 0}$$

$$f(0) - 0 + 0 = c$$

$$0 = c$$

$$f(x) = \eta\mu x - x\,\sigma\upsilon\nu x$$

6. $f(0)=0$ $\qquad$ $x\big(2f(x)+xf'(x)\big)=1-f'(x)$

Nδο $\quad f(x)=\dfrac{x}{x^2+1}$

$$(x^2+1)\,f(x)=x$$

$$\underbrace{(x^2+1)\,f(x)-x}_{\varphi(x)}=0$$

$$\boxed{\varphi'(x)=2x\,f(x)+(x^2+1)\,f'(x)-1}$$

$$2x\,f(x)+x^2f'(x)-1+f'(x)=0$$

$$2x\,f(x)+(x^2+1)-1=0$$

$$\varphi'(x)=0$$

$$\varphi(x)=c$$

Αρχ

$$\varphi(x)=c$$

$$(x^2+1)\,f(x)-x=c$$

$$x=0$$

$$f(0)-0=c$$

$$c=0$$

$$f(x)=\frac{x}{x^2+1}$$

Ενότητα 22

9.

$f(0)=0$

$$x\left(f(x)+x f'(x)\right)=\sqrt{x^2+1}-f'(x)$$

Νδο $f(x)=\dfrac{x}{\sqrt{x^2+1}}$

$$f(x)\sqrt{x^2+1}=x$$

$$\underbrace{f(x)\sqrt{x^2+1}-x}_{g(x)}=0$$

$$g'(x)=f'(x)\sqrt{x^2+1}+f(x)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}-1$$

$$g'(x)=\frac{f'(x)(x^2+1)+xf(x)-\sqrt{x^2+1}}{\sqrt{x^2+1}}=0$$

$\Rightarrow g(x)=c$

$$xf(x)+x^2f'(x)+f'(x)-\sqrt{x^2+1}=0$$

$$xf(x)+(x^2+1)f'(x)-\sqrt{x^2+1}=0.$$

Αφού $g(x) = c$

$$f(x)\sqrt{x^2+1} - x = c$$

$\underline{x = 0}$

$$f(0) - 0 = c$$

$$\underline{\underline{c = 0}}$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

II. $f(1)=0$

$$f'(x) = f(x) + \frac{e^x}{x}$$

Ndo $f(x) = e^x \ln x$

$$\underbrace{f(x) - e^x \ln x}_{g(x)} = 0$$

$$g'(x) = f'(x) - e^x \ln x - e^x \frac{1}{x}$$

$$g'(x) = f(x) + \cancel{\frac{e^x}{x}} - e^x \ln x - \cancel{\frac{e^x}{x}}$$

$$g'(x) = f(x) - e^x \ln x$$

$$g'(x) = g(x)$$

$$\Rightarrow g(x) = ce^x$$

$$g(x) = ce^x$$

$$f(x) - e^x \ln x = ce^x$$

$\underline{x=1}$

$$f(1) = ce$$

$$0 = ce$$

$$c = 0.$$

$$f(x) = e^x \ln x$$

Απόδειξη

$$g'(x) = g(x)$$

$$e^x g'(x) = e^x g(x)$$

$$e^x g'(x) - e^x g(x) = 0$$

$$\frac{e^x g'(x) - e^x g(x)}{e^{2x}} = 0$$

$$\left(\frac{g(x)}{e^x}\right)' = 0$$

$$\frac{g(x)}{e^x} = c$$

$$g(x) = ce^x$$

12. $f'(x) = 3f(x) + 2x\,e^{3x}$ $\quad f(0) = 0$

Nδο $f(x) = x^2 e^{3x}$

$$\underbrace{f(x) - x^2 e^{3x}}_{g(x)} = 0$$

$$g'(x) = f'(x) - 2xe^{3x} - x^2 3e^{3x}$$

$$g'(x) = 3f(x) + \cancel{2xe^{3x}} - \cancel{2xe^{3x}} - 3x^2 e^{3x}$$

$$g'(x) = 3f(x) - 3x^2 e^{3x}$$

$$g'(x) = 3\left(f(x) - x^2 e^{3x}\right)$$

$$g'(x) = 3g(x)$$

$$g'(x) - 3g(x) = 0$$

Δεν δούλεψε η μέθοδος.

$$f'(x) = 3f(x) + 2xe^{3x}$$

$$\boxed{f'(x) - 3f(x) = 2xe^{3x}}$$

$$g(x) = -3$$

$$G(x) = -3x$$

[illegible] [illegible] $e^{G(x)}$ [illegible] [illegible] $\underline{\underline{e^{-3x}}}$

$$e^{-3x}f'(x) - 3e^{-3x}f(x) = 2xe^{3x}e^{-3x}$$

$$\left(e^{-3x}f(x)\right)' = 2x$$

$$\left(e^{-3x}f(x)\right)' = \left(x^2\right)'$$

$$e^{-3x}f(x) = x^2 + c$$

$$\underline{x = 0}$$

$$f(0) = c \Rightarrow c = 0$$

$$e^{-3x} f(x) = x^2$$

$$\frac{1}{e^{3x}} f(x) = x^2$$

$$f(x) = x^2 e^{3x}$$

$$f'(x) + g(x)\cdot f(x) = 0$$

Εστω $G(x)$ παραγουσα της $g(x)$

$$\Rightarrow G'(x) = g(x)$$

Πολ/ζω παντου με το $e^{G(x)}$

$$e^{G(x)} f'(x) + g(x) e^{G(x)} f(x) = 0$$

$$\left(e^{G(x)} f(x)\right)' = 0$$

$$\boxed{e^{G(x)} f(x) = c}$$

B' τρόπος

$$f(x) = x^2 e^{3x}$$

$$\frac{f(x)}{e^{3x}} = x^2$$

$$\underbrace{\frac{f(x)}{e^{3x}} - x^2}_{g(x)} = 0$$

$$g'(x) = \frac{f'(x)\cancel{e^{3x}} - f(x)3\cancel{e^{3x}}}{(e^{3x})^{\cancel{2}}} - 2x$$

$$g'(x) = \frac{f'(x) - 3f(x)}{e^{3x}} - 2x$$

$$g'(x) = \frac{\cancel{3f(x)} + 2xe^{3x} - \cancel{3f(x)}}{e^{3x}} - 2x$$

$$g'(x) = 2x - 2x = 0$$

ΕΤΣ.

13. $f(0) = \ln 2$

$$f'(x) = e^{x - f(x)}$$

Νδο $f(x) = \ln(1+e^x)$

$$e^{f(x)} = e^x + 1$$

$$f(x) = \ln(e^x + 1)$$

$$f'(x) = \frac{e^x}{e^{f(x)}}$$

$$f'(x)\, e^{f(x)} = e^x$$

$$\left(e^{f(x)}\right)' = \left(e^x\right)'$$

$$e^{f(x)} = e^x + c$$

$\underline{x=0}$

$$e^{f(0)} = 1 + c$$

$$e^{\ln 2} = 1 + c \Rightarrow 2 = 1 + c \Rightarrow \underline{\underline{c=1}}$$

$$f(x) = \ln(1+e^x)$$

$$e^{f(x)} = 1+e^x$$

$$\underbrace{e^{f(x)} - 1 - e^x}_{g(x)} = 0$$

$$g'(x) = f'(x)\, e^{f(x)} - e^x$$

$$g'(x) = e^{x-f(x)}\, e^{f(x)} - e^x$$

$$g'(x) = e^x - e^x = 0$$

17. $f:(0,\pi) \longrightarrow R$

$$f'(x) = f(x)(1-\sigma\varphi x).$$

(α) Αν $g(x) = f(x)\,\eta\mu x$ νδο $g(x) = ce^{x}$

$$g'(x) = f'(x)\,\eta\mu x + f(x)\,\sigma\upsilon\nu x$$

$$g'(x) = f(x)(1-\sigma\varphi x)\,\eta\mu x + f(x)\,\sigma\upsilon\nu x$$

$$g'(x) = f(x)\,\eta\mu x - f(x)\,\eta\mu x\,\sigma\varphi x + f(x)\,\sigma\upsilon\nu x$$

$$g'(x) = f(x)\,\eta\mu x - f(x)\cancel{\eta\mu x}\,\frac{\sigma\upsilon\nu x}{\cancel{\eta\mu x}} + f(x)\,\sigma\upsilon\nu x$$

$$g'(x) = f(x)\,\eta\mu x \cancel{- f(x)\,\sigma\upsilon\nu x} \cancel{+ f(x)\,\sigma\upsilon\nu x}$$

$$g'(x) = f(x)\,\eta\mu x$$

$$g'(x) = g(x)$$

$$g(x) = ce^{x}.$$

13 Αν $f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}$

Αφου $g(x) = ce^{x}$

$$f(x)\,\eta\mu x = ce^{x}$$

$$x = \frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right)\eta\mu\frac{\pi}{2} = ce^{\frac{\pi}{2}}$$

$$\cancel{e^{\frac{\pi}{2}}} \cdot 1 = c\cancel{e^{\frac{\pi}{2}}}$$

$$c = 1$$

$$f(x) = \frac{e^{x}}{\eta\mu x}$$

15. $f''(x) = g''(x)$

$f'(1) = g'(1)$

$f(0) - g(0) = 3$

Nδο $f(1) = g(1) + 3$

$$f'(x) = g'(x) + c$$

$$\underline{x = 1}$$

$$\cancel{f'(1)} = \cancel{g'(1)} + c$$

$$c = 0$$

$$f'(x) = g'(x)$$

$$f(x) = g(x) + c$$

$$\underline{x = 0}$$

$$f(0) = g(0) + c$$

$$f(0) - g(0) = c$$

$$3 = c$$

$$f(x) = g(x) + 3$$

$$\underline{x = 1}$$

$$f(1) = g(1) + 3.$$

24. $f(x) - 2f'(x) + f''(x) = 2e^x$

Θέλω $g(x) = 2x - \dfrac{f'(x) - f(x)}{e^x}$ σταθερή

$$g'(x) = 2 - \frac{(f''(x) - f'(x))\cancel{e^x} - (f'(x) - f(x))\cancel{e^x}}{(e^x)^{\cancel{2}}}$$

$$g'(x) = 2 - \frac{f''(x) - f'(x) - f'(x) + f(x)}{e^x}$$

$$g'(x) = 2 - \frac{f''(x) - 2f'(x) + f(x)}{e^x}$$

$$g'(x) = \frac{2e^x - f''(x) + 2f'(x) - f(x)}{e^x}$$

$g'(x) = 0.$

$g(x) = c.$

$$2x - \frac{f'(x) - f(x)}{e^x} = C$$

$$\underline{x = 0}$$

$$\cancel{- \frac{f'(0) - f(0)}{1}} = C$$

$$C = 0.$$

$$2x - \frac{f'(x) - f(x)}{e^x} = 0$$

$$2x = \frac{f'(x) - f(x)}{e^x}$$

$$(x^2)' = \frac{e^x f'(x) - e^x f(x)}{(e^x)^2}$$

$$(x^2)' = \left(\frac{f(x)}{e^x}\right)'$$

$$x^2 = \frac{f(x)}{e^x} + C$$

$$x = 0$$

$$0 = \frac{f(0)}{1} + C$$

$$C = 0$$

$$x^2 = \frac{f(x)}{e^x}$$

$$f(x) = x^2 e^x$$

Ενότητα 26

28. (α) $f(x) = e^x + \frac{x^3}{6} - \frac{x^2}{2} - ημx$

$f'(x) = e^x + \frac{1}{6}3x^2 - \frac{1}{2}2x - συνx$

$\boxed{f'(x) = e^x + \frac{1}{2}x^2 - x - συνx}$ $\quad f'(0)=0$

$f''(x) = e^x + x - 1 + ημx \quad f''(0)=0$

$f'''(x) = \underset{+}{e^x} + \underbrace{1 + συνx}_{+} > 0,$

$-1 \le συνx \le 1$

$0 \le 1+συνx \le 2$

x		0	
f'''	$+$		$+$
f''	$\nearrow$ $-$	0	$\nearrow$ $+$
f'	$+$ $\searrow$	0	$+$ $\nearrow$
f	$\nearrow$		$\nearrow$

$x<0 \Rightarrow f''(x)<f''(0) \Rightarrow f''(x)<0$

$x>0 \Rightarrow f''(x)>f''(0)$

$f''(x)>0$

$f'(x) \geqslant f'(0)$

$f'(x) \geqslant 0$

28. B $f(x) = \frac{\ln x}{x-2}$

$$D_f = (0,2) \cup (2,+\infty)$$

$$f'(x) = \frac{\frac{1}{x}(x-2) - \ln x}{(x-2)^2}$$

$$\boxed{f'(x) = \frac{x-2-x\ln x}{x(x-2)^2}}$$

$$\varphi(x) = x-2-x\ln x$$

$$\varphi'(x) = \cancel{1} - \ln x - \cancel{1}$$

$$\varphi'(x) = -\ln x$$

$\rightarrow \varphi'(x) = 0 \Rightarrow -\ln x = 0$

$x = 1$

x	0		1		2	
φ'		$+$	0	$-$	||	$-$
φ		$\nearrow$	0	$\searrow$	||	$\searrow$ $-$
$x(x-2)^2$		$+$		$+$	||	$+$
f'		$-$		$-$	||	$-$
f		$\searrow$		$\searrow$	||	$\searrow$

$\varphi'(x) \leq \varphi'(1)$

$\varphi'(x) \leq 0$

$x > 2$

$\varphi(x) < \varphi(2)$

$\varphi(x) < -2\ln 2$

Επορενω Μαθημα

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