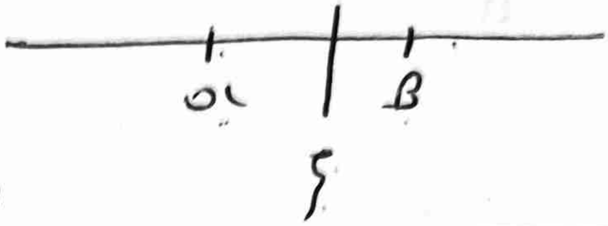
Ενοτητα 21

6. ⓐ $e^{\alpha} < \frac{e^{\beta}-e^{a}}{\beta-\alpha} < e^{\beta}$

$\boxed{f(x)=e^{x}}$



$$f'(\xi)=\frac{f(\beta)-f(a)}{\beta-a}=\frac{e^{\beta}-e^{a}}{\beta-a}$$

$f(x)=e^{x}$

$f'(x)=e^{x}$

$f''(x)=e^{x}>0$

$f' \nearrow$

$$\alpha < \xi < \beta$$

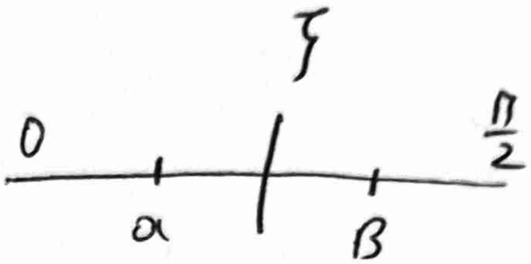
$$f'$$

$$f'(a) < f'(\xi) < f'(\beta)$$

$$\boxed{e^{a} < \frac{e^{\beta}-e^{a}}{\beta-a} < e^{\beta}}$$

③ $(\beta-\alpha)\,\text{συν}\,\beta < \text{ημ}\,\beta - \text{ημ}\,\alpha < (\beta-\alpha)\,\text{συν}\,\alpha$

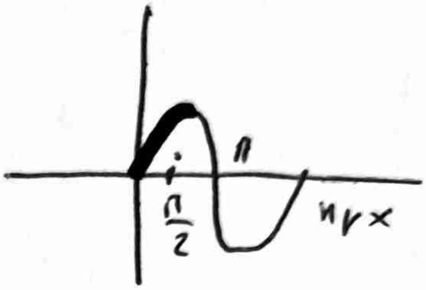
$f(x) = \text{ημ}\,x$



$f'(x) = \text{συν}\,x$

$f''(x) = -\underset{\oplus}{\text{ημ}\,x} < 0$

$f' \downarrow$



$$f'(\xi) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$$

$$f'(\xi) = \frac{\text{ημ}\,\beta - \text{ημ}\,\alpha}{\beta-\alpha}$$

$$\alpha < \xi < \beta$$

$$f' \downarrow$$

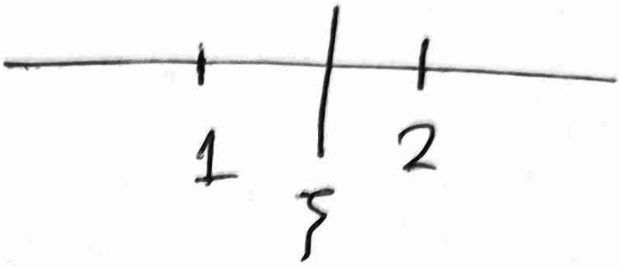
$$f'(\alpha) > f'(\xi) > f'(\beta)$$

$$\text{συν}\,\alpha > \frac{\text{ημ}\,\beta - \text{ημ}\,\alpha}{\beta-\alpha} > \text{συν}\,\beta$$

$(\beta-\alpha)\,\text{συν}\,\alpha > \text{ημ}\,\beta - \text{ημ}\,\alpha > (\beta-\alpha)\,\text{συν}\,\beta.$

17. $f'(x) < \ln 2$ $\quad f(1) = e\ln 2$

Νδο $f(2) < \ln 2^{e+1}$



$$f'(\xi) = \frac{f(2) - f(1)}{2 - 1}$$

Αφού $f'(x) < \ln 2$

$$f'(\xi) < \ln 2$$

$$\frac{f(2) - f(1)}{2 - 1} < \ln 2$$

$$f(2) - e\ln 2 < \ln 2$$

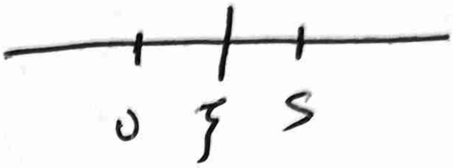
$$f(2) < e\ln 2 + \ln 2$$

$$f(2) < (e+1)\ln 2$$

$$f(2) < \ln 2^{e+1}$$

16. $f(0)=1$ $\qquad 2 \leq f'(x) \leq 4$

Νδο $11 \leq f(s) \leq 21$



$$f'(\xi) = \frac{f(s)-f(0)}{s-0} = \frac{f(s)-1}{s}$$

όμως γνωρίζω ότι $\quad 2 \leq f'(x) \leq 4$

$$2 \leq f'(\xi) \leq 4$$

$$2 \leq \frac{f(s)-1}{s} \leq 4$$

$$10 \leq f(s)-1 \leq 20$$

$$11 \leq f(s) \leq 21$$

20.

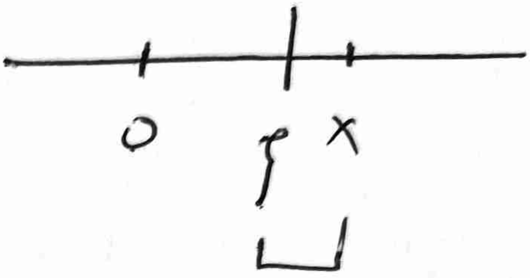
$f' \nearrow$

$f(0)=0$

$f'(x)>1$ $\forall x>0.$

Νδο $x < f(x) < x f'(x)$

$$f'(\xi) = \frac{f(x)-f(0)}{x-0}$$



$$f'(\xi) = \frac{f(x)-0}{x} = \frac{f(x)}{x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x) \Rightarrow \frac{f(x)}{x} < f'(x)$$

$$f(x) < x f'(x)$$

Άρα $f'(x)>1$ $\forall x>0$

$$f'(\xi)>1$$

$$\frac{f(x)}{x}>1$$

$$f(x)>x$$

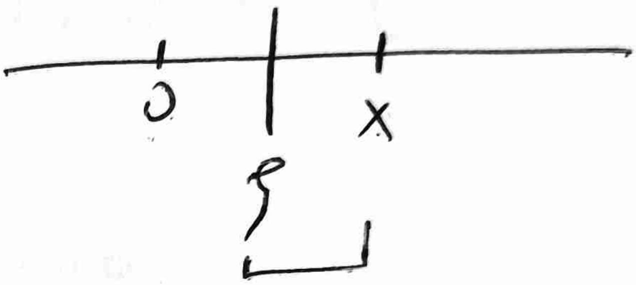
21. Η $f(x)$ εφαπτεται στον $x'x$. στην αρχη των αξονων.

$f' \nearrow$ $\Longrightarrow f'(0)=0$

$f(0)=0$.

(α) Ν.δ.ο. $x f'(x) > f(x)$ $\forall x>0$

$$f'(\xi)=\frac{f(x)-f(0)}{x-0}$$



$$f'(\xi)=\frac{f(x)}{x}$$

$\xi<x \Longrightarrow f'(\xi)<f'(x) \Longrightarrow \frac{f(x)}{x}<f'(x)$

$$\underline{\underline{f(x)<xf'(x)}}$$

(β) $\frac{f(\beta)-\beta f'(\beta)}{x-2}-\frac{f'(\beta)}{x-1}=0$ $\beta>0$

Εχει ριζα $(1,2)$,

$$\underbrace{\boxed{(x-1)\left(f(B)-Bf'(B)\right)-(x-2)f'(B)=0}}_{g(x)}$$

$$\left.\begin{array}{l} g(1)=f'(B)>0 \\ \\ g(2)=f(B)-Bf'(B)<0 \end{array}\right\} g(1)g(2)<0 \ \checkmark$$

- $x f'(x) > f(x) \quad \forall x > 0$

  $B f'(B) > f(B)$

  $$\underline{\underline{0 > f(B) - Bf'(B)}}$$

- $B > 0$

  $f'(B) > f'(0)$

  $$\underline{f'(B) > 0}$$

23. $|f'(x)| \geq 1 \quad \forall x \in [1,3]$

$$\left.\begin{array}{l} A(1, f(1)) \\ B(3, f(3)) \end{array}\right\} \text{νδο } (AB) \geq 2\sqrt{2}$$

$$(AB) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$(AB) = \sqrt{(f(3) - f(1))^2 + (3-1)^2}$$

$$\boxed{(AB) = \sqrt{(f(3) - f(1))^2 + 4}}$$

Αρκει νδο $\sqrt{(f(3)-f(1))^2 + 4} \geq 2\sqrt{2}$

$$f'(\xi) = \frac{f(3) - f(1)}{3-1}$$

$$f'(\xi) = \frac{f(3) - f(1)}{2}$$

$$|f'(x)| \geq 1$$

$$|f'(\xi)| \geq 1$$

$$\left| \frac{f(3)-f(1)}{2} \right| \geq 1$$

$$\frac{|f(3)-f(1)|}{|2|} \geq 1$$

$$|f(3)-f(1)| \geq 2$$

$$(f(3)-f(1))^2 \geq 4$$

$$(f(3)-f(1))^2+4 \geq 8$$

$$\sqrt{(f(3)-f(1))^2+4} \geq \sqrt{8}$$

$$\underline{AB \geq 2\sqrt{2}}$$

32. $f(-1) = -1$ $f(1) = 1$

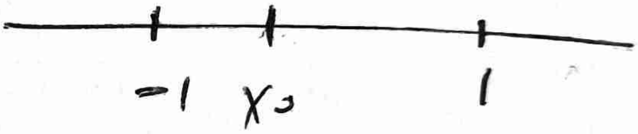
ⓐ Νδο $\exists x_0 \in (-1,1)$ τ.ω $f(x_0) = 0$

$f(-1) \cdot f(1) = -1 < 0$

Bolzano $\exists x_0 \in (-1,1)$ τ.ω $f(x_0) = 0$,

Ⓑ Νδο $\exists \xi_1, \xi_2 \in (-1,1)$ τ.ω $\frac{1}{f'(\xi_1)} + \frac{1}{f'(\xi_2)} = 2$

$$f'(\xi_1) = \frac{f(x_0) - f(-1)}{x_0 + 1}$$



$$f'(\xi_2) = \frac{f(1) - f(x_0)}{1 - x_0}$$

$$\frac{1}{f'(\xi_1)} + \frac{1}{f'(\xi_2)} = \frac{1}{\frac{0-(-1)}{x_0+1}} + \frac{1}{\frac{1-0}{1-x_0}} =$$

$$= \cancel{x_0} + 1 + 1 - \cancel{x_0} = 2,$$

25. (α) Νδο $|ημ\sqrt{x^2+1} - ημx| \le |\sqrt{x^2+1} - x|$

Ισχυει οτι $\sqrt{x^2+1} > x$

- Αν $x<0$ ισχυει.
- Αν $x=0$ $1>0$ ισχυει.
- Αν $x>0$ $\sqrt{x^2+1} > x$

$$\cancel{\sqrt{}}{x^2+1}^{\cancel{2}} > x^2$$

$$\cancel{x^2}+1 > \cancel{x^2}$$

$$1>0 \checkmark$$

$\forall x \in R$

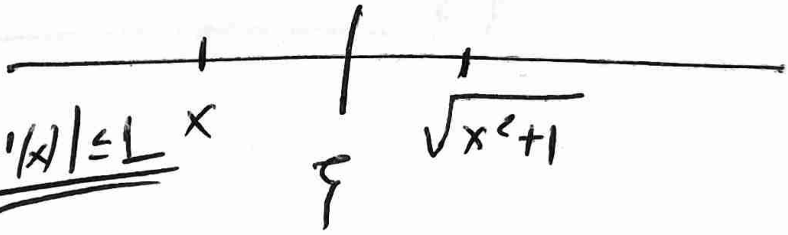
η ανισοτητα ισχυει.

$f(x)=ημx$

$f'(x)=συνx$

$-1 \le συνx \le 1$

$|συνx| \le 1 \Rightarrow \underline{\underline{|f'(x)| \le 1}}$



$$f'(ξ) = \frac{f(\sqrt{x^2+1}) - f(x)}{\sqrt{x^2+1} - x}$$

Αφού $|f'(x)| \leq 1$

$$|f'(\xi)| \leq 1$$

$$\left| \frac{f(\sqrt{x^2+1}) - f(x)}{\sqrt{x^2+1} - x} \right| \leq 1$$

$$\left| \frac{ημ\sqrt{x^2+1} - ημx}{\sqrt{x^2+1} - x} \right| \leq 1$$

$$\frac{|ημ\sqrt{x^2+1} - ημx|}{|\sqrt{x^2+1} - x|} \leq 1$$

$$|ημ\sqrt{x^2+1} - ημx| \leq |\sqrt{x^2+1} - x|,$$

③ $\lim_{x\to+\infty} \left(ημ\sqrt{x^2+1} - ημx \right)$

$$\left| ημ\sqrt{x^2+1} - ημx \right| \leq \left| \sqrt{x^2+1} - x \right|$$

$$\boxed{-\left|\sqrt{x^2+1}-x\right| \leq ημ\sqrt{x^2+1} - ημx \leq \left|\sqrt{x^2+1}-x\right|}$$

$$\to \lim_{x\to+\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to+\infty} \frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to+\infty} \frac{\cancel{\sqrt{x^2+1}}^{2} - \cancel{x^2}}{\sqrt{x^2+1}+x} = 0$$

- $\lim_{x\to+\infty} -\left|\sqrt{x^2+1}-x\right| = 0$
- $\lim_{x\to+\infty} \left|\sqrt{x^2+1}-x\right| = 0$

} Άρω Κ.Π

$\lim_{x\to+\infty} ημ\sqrt{x^2+1} - ημx = 0$

Συνεπειες ΘΜΤ

1. Θεωρημα σταθερης συναρτησης

Αν η f ορισμενη στο Δ

Αν η f συνεχης στο Δ

Αν $f'(x)=0$ $\forall x$ εσωτερικο του Δ.

Τοτε η $f(x)$ ειναι σταθερη σε ολο το Δ. $\Rightarrow$ $f(x) = c$

2. Αν f, g συνεχείς στο Δ.

Αν $f'(x) = g'(x)$ $\forall x$ εσωτερικό του Δ

Τότε $f(x) = g(x) + c$.

3. Αν $f'(x) = f(x)$

Τότε $f(x) = ce^{x}$

αυτό θέλει απόδειξη

Ενοτητα 22

4. $f: R \to R$

$f'(x) = f(x) + 2xe^x$

ⓐ Nδο $g(x) = \frac{f(x)}{e^x} - x^2$ σταθερη.

$$g'(x) = \frac{f'(x)\,e^x - f(x)\,e^x}{(e^x)^2} - 2x$$

$$g'(x) = \frac{f'(x) - f(x)}{e^x} - 2x$$

$$g'(x) = \frac{\cancel{f(x)} + 2xe^x \cancel{-f(x)}}{e^x} - 2x$$

$$g'(x) = \frac{2x\cancel{e^x}}{\cancel{e^x}} - 2x$$

$$g'(x) = 2x - 2x$$

$$g'(x) = 0, \Rightarrow g(x) = c.$$

Ⓑ Αν $f(0)=1$ Βρε τύπο f.

$$g(x)=c$$

$$\frac{f(x)}{e^x}-x^2=c$$ 6

$$\underline{x=0}$$

$$\frac{f(0)}{e^0}-0^2=c$$

$$\frac{1}{1}-0=c$$

$$\underline{\underline{c=1}}$$

$$\frac{f(x)}{e^x}-x^2=1 \Rightarrow \frac{f(x)}{e^x}=x^2+1$$

$$\boxed{f(x)=e^x(x^2+1)}$$

Πανελληνια 2025

- $f:(0,+\infty)\longrightarrow \mathbb{R}$ παρ/μη
- F παραγουσα της f $\Rightarrow F'(x)=f(x)$
- $x\,f(x) = 2F(x)\ln x \quad \forall x>0$
- Η εφαπτομενη της f στο $M(1,f(1))$ ειναι παραλληλη στην ευθεια $y=2x$

Δ1: Νδο $g(x)=\dfrac{F(x)}{x^{\ln x}}$, $x>0$ σταθερη.

$$g'(x)=\frac{f(x)\,x^{\ln x}-F(x)\left(x^{\ln x}\right)'}{\left(x^{\ln x}\right)^2}$$

$$g'(x)=\frac{f(x)\,x^{\ln x}-F(x)\,x^{\ln x}\,\frac{2\ln x}{x}}{\left(x^{\ln x}\right)^2}$$

$$g'(x)=\frac{f(x)-F(x)\,\frac{2\ln x}{x}}{x^{\ln x}}$$

$$x^{\ln x} = e^{\ln x^{\ln x}} = e^{\ln x \cdot \ln x} = e^{\ln^2 x}$$

$$(x^{\ln x})' = (e^{\ln^2 x})' = e^{\ln^2 x} (\ln^2 x)' =$$

$$= e^{\ln^2 x} \cdot 2 \ln x \frac{1}{x}$$

$$\boxed{(x^{\ln x})' = x^{\ln x} \frac{2 \ln x}{x}}$$

$$g'(x) = \frac{x f(x) - 2\ln x \, F(x)}{x \cdot x^{\ln x}} .$$

$$g'(x) = \frac{\cancel{2F(x)\ln x} - \cancel{2\ln x F(x)}}{x \cdot x^{\ln x}}$$

$$g'(x) = 0 \Rightarrow g(x) = c ,$$

Δ2: i) Να υπολογίσετε το $\lim\limits_{x \to 1} \frac{f(x)}{\ln x}$

- $x f(x) = 2F(x) \ln x$

$\underline{x = 1}$

$\boxed{f(1) = 0}$

$$\longrightarrow \lim_{x \to 1} \frac{f(x)}{\ln x} \overset{\left(\frac{0}{0}\right)}{\underset{DLH}{=}} \lim_{x \to 1} \frac{f'(x)}{\frac{1}{x}} = \frac{2}{1} = 2 .$$

Η εφαπτομενη τη f στο $M(1, f(1))$

ειναι παραλληλη $y = 2x$

$$y - f(1) = f'(1)(x-1) \parallel y = 2x$$

$$\Rightarrow \boxed{f'(1) = 2}$$

f συν + παρ

f'

Ολο αυτο το εκανα για νδο η $f'(x)$ συνεχη για να μπορω να αντικαταστησω το $f'(1)$.

- $x f(x) = 2F(x) \ln x$

$$f(x) = \frac{2F(x) \ln x}{x}$$

$$f'(x) = \frac{(2F(x)\ln x)' x - 2F(x) \ln x \cdot 1}{x^2}$$

$$f'(x) = \frac{2f(x) \ln x + 2F(x) \frac{1}{x} - 2F(x) \ln x}{x^2}$$ παρ με συνεχ.

B' τροπος

$$\lim_{x\to1}\frac{f(x)}{\ln x}=\lim_{x\to1}\frac{\frac{f(x)-f(1)}{x-1}}{\frac{\ln x}{x-1}}=\frac{f'(1)}{1}=2.$$

$$\to \lim_{x\to1}\frac{\ln x}{x-1}=\lim_{x\to1}\frac{1}{x}=1$$

11). Νδο $F(1)=1$ και $F(x) = x^{\ln x}$

- $x f(x) = 2F(x)\ln x$

$$F(x) = \frac{x f(x)}{2\ln x}$$

$$F(1) = \lim_{x\to 1} \frac{x f(x)}{2\ln x} = \frac{1\cdot 2}{2} = 1$$

$$= \lim_{x\to 1}$$

$F(1)=1$

$$\boxed{F(1) = \lim_{x\to 1} F(x)}$$

F συνεχής

Εδειξα ότι η $g(x)$ σταθερή

$$g(x) = c \Rightarrow \frac{F(x)}{x^{\ln x}} = c$$

$x=1$

$$\frac{F(1)}{1} = c$$

$c=1$

$$F(x) = x^{\ln x}$$

Δ3: Μονοτονια της $F(x)$ και

επιλυση εξισωσης $F(x^2) = F(x) - (x-1)^2$

στο $(0,+\infty)$.

$$F(x) = x^{\ln x}$$

$$F'(x) = f(x) = x^{\ln x} \frac{2\ln x}{x}, \quad x > 0$$

$$\rightarrow F'(x) = 0 \Rightarrow \frac{2\ln x}{x} = 0 \Rightarrow \underline{\underline{x = 1}}$$

x	0	1
F'	−	0 +
F	↘	↗

$$F(x) \geq F(1)$$

$$\underline{\underline{F(x) \geq 1}}$$

$F(x^2) = F(x) - (x-1)^2$ $(0,+\infty)$.

$F(x^2) - F(x) + (x-1)^2 = 0$ Προφανής ρίζα $x=1$

$\underline{0<x<1}$

• $x^2<x \Rightarrow F(x^2) > F(x) \Rightarrow F(x^2)-F(x) > 0$

$$\underbrace{F(x^2)-F(x)}_{\oplus} + \underbrace{(x-1)^2}_{\oplus} = 0$$ Αδύνατη.

$\underline{\underline{x=1}}$ $\underline{\underline{x=1}}$

$\underline{x>1}$

• $x^2>x \Rightarrow F(x^2) > F(x) \Rightarrow F(x^2)-F(x) > 0$

$$\underbrace{F(x^2)-F(x)}_{\oplus} + (x-1)^2 = 0$$ $\oplus$ Αδύνατη

$x=1$ $x=1$

Προφανής $\underline{x=1}$

7. $f: (0, +\infty) \to R$ $\quad f(1) = 1$

παρ/μη

$$x^2 f'(x) + x f(x) = 1.$$

Νδο $f(x) = \frac{1 + \ln x}{x}$, $x > 0$.

$$x f(x) = 1 + \ln x$$

$$\underbrace{x f(x) - 1 - \ln x}_{g(x)} = 0$$

$$g'(x) = f(x) + x f'(x) - \frac{1}{x}$$

$$\boxed{g'(x) = \frac{x f(x) + x^2 f'(x) - 1}{x} = 0}$$

$$g(x) = c$$

$$x f(x) - 1 - \ln x = c \quad \longrightarrow \quad \boxed{f(x) = \frac{\ln x + 1}{x}}$$

$$\underline{x = 1}$$

$$f(1) - 1 - 0 = c$$

$$1 - 1 = c \quad c = 0$$

Επομενο Μαθημα

22

②

③

⑤

⑥