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Ενότητα 35

(1) $\int_0^1 x\sqrt{2-x^2}\,dx \overset{(*)}{=}$

Θέτω $\sqrt{2-x^2}=t$

$2-x^2=t^2$

$-2x\,dx=2t\,dt$

$-x\,dx=t\,dt$

$x\,dx=-t\,dt$

$x=1 \quad t=1$

$x=0 \quad t=\sqrt2$

$\overset{(*)}{=} \int_{\sqrt2}^{1} -t\cdot t\,dt = -\int_{\sqrt2}^{1} t^2\,dt$

$= \int_1^{\sqrt2} t^2\,dt = \left(\frac{t^3}{3}\right)_1^{\sqrt2} = \frac13\left(t^3\right)_1^{\sqrt2} =$

$= \frac13\left(\sqrt2^{\,3}-1\right) = \frac13\left(2\sqrt2-1\right)$

Σχόλιο

$\int_\alpha^\beta f(x)\,dx = -\int_\beta^\alpha f(x)\,dx$

(ε) $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2-1}{u}\,2u\,du$

$\sqrt{x+1} = u$

$x+1 = u^2$

$x = u^2 - 1$

$dx = 2u\,du$

$= \int_1^2 (u^2-1)\,2\,du$

$= 2\int_1^2 u^2-1\,du$

$= 2\int_1^2 u^2\,du - 2\int_1^2 1\,du$

$= 2\left(\frac{u^3}{3}\right)_1^2 - 2\,(u)_1^2$

$= \frac{2}{3}\,(u^3)_1^2 - 2\,(2-1)$

$= \frac{2}{3}\,7 - 2 = \frac{14}{3} - \frac{6}{3} = \frac{8}{3}.$

17. B $\int_{-1}^{0} x^2 (x-1)^4 \, dx =$

$$= \int_{-2}^{-1} (t+1)^2 \cdot t^4 \, dt$$

$x-1 = t$

$x = t+1$

$dx = dt$

$$= \int_{-2}^{-1} (t^2+2t+1) \, t^4 \, dt$$

$$= \int_{-2}^{-1} t^6 + 2t^5 + t^4 \, dt$$

$$= \frac{1}{7}\left(t^7\right)_{-2}^{-1} + \frac{2}{6}\left(t^6\right)_{-2}^{-1} + \frac{1}{5}\left(t^5\right)_{-2}^{-1}$$

16. (δ) $\int_1^{\frac{1}{2}} \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\int_1^2 e^t\,dt$

$\frac{1}{x} = t$

$-\frac{1}{x^2}\,dx = dt$

$= -\left(e^t\right)_1^2 =$

$= -\left(e^2 - e\right) = e - e^2$

(ε) $\int_0^{\pi/2} \text{συν}x \cdot \text{συν}(\text{ημ}x)\,dx = \int_0^1 \text{συν}t\,dt$

$\text{ημ}x = t$

$\text{συν}x\,dx = dt$

$= \left(\text{ημ}t\right)_0^1 =$

$= \text{ημ}1$

(στ) $\int_e^{e^2} -\frac{dx}{x\sqrt{\ln x}} = -\int_1^{\sqrt{2}} \frac{1}{\cancel{t}}\, 2\cancel{t}\,dt =$

$\sqrt{\ln x} = t$

$\ln x = t^2$

$\frac{1}{x}\,dx = 2t\,dt$

$= -2\int_1^{\sqrt{2}} 1\,dt =$

$= -2\left(t\right)_1^{\sqrt{2}} = -2\left(\sqrt{2}-1\right)$

15. δ $\int_0^1 \frac{1}{(3x+1)^2}\,dx = \int_1^4 \frac{1}{t^2}\,\frac{1}{3}\,dt$

$3x+1 = t$

$3\,dx = dt$

$dx = \frac{1}{3}\,dt$

$= \frac{1}{3}\int_1^4 \frac{1}{t^2}\,dt$

$= \frac{1}{3}\left(-\frac{1}{t}\right)_1^4 =$

$= -\frac{1}{3}\left(\frac{1}{4}-1\right) = -\frac{1}{12}+\frac{1}{3}$

ε $\int_0^4 \sqrt{2x+1}\,dx = \int_1^3 u\,u\,du$

$\sqrt{2x+1} = u$

$2x+1 = u^2$

$2\,dx = 2u\,du$

$dx = u\,du$

$= \int_1^3 u^2\,du$

$= \frac{1}{3}\left(u^3\right)_1^3$

$= \frac{1}{3}(27+1)$

13. (γ) Νδο $\int_1^3 f\left(\frac{3}{x}\right) dx = 3\int_1^3 \frac{f(x)}{x^2} dx$

$$\int_1^3 f\left(\frac{3}{x}\right) dx = -\int_3^1 f(u)\frac{3}{u^2}\,du$$

$$= 3\int_1^3 \frac{f(u)}{u^2}\,du$$

$$= 3\int_1^3 \frac{f(x)}{x^2}\,dx$$

$$\frac{3}{x} = u$$

$$-\frac{3}{x^2}\,dx = du$$

$$-\frac{3}{x}\cdot\frac{3}{x}\cdot\frac{1}{3}\,dx = du$$

$$-u\cdot u\cdot\frac{1}{3}\,dx = du$$

$$-u^2\,\frac{1}{3}\,dx = du$$

$$-\frac{1}{3}\,dx = \frac{1}{u^2}\,du$$

$$dx = -\frac{3}{u^2}\,du$$

14. $f(x) = x^4 - x^3 + \lambda x^2 - \lambda x$

$$\int_a^1 x\, f(x^2)\, dx = 0 \quad \Rightarrow \int_0^1 f(t)\, \frac{1}{2}\, dt = 0$$

$x^2 = t$

$2x\, dx = dt$

$x\, dx = \frac{1}{2}\, dt$

$$\frac{1}{2} \int_0^1 f(t)\, dt = 0$$

$$\int_0^1 f(t)\, dt = 0$$

$$\int_0^1 f(x)\, dx = 0$$

$$\int_0^1 x^4 - x^3 + \lambda x^2 - \lambda x \quad dx = 0$$

$$\int_0^1 x^4\, dx - \int_0^1 x^3\, dx + \int_0^1 \lambda x^2\, dx - \int_0^1 \lambda x\, dx = 0$$

$$\frac{1}{5} \left(x^5\right)\Big|_0^1 - \frac{1}{4} \left(x^4\right)\Big|_0^1 + \frac{\lambda}{3} \left(x^3\right)\Big|_0^1 - \frac{\lambda}{2} \left(x^2\right)\Big|_0^1 = 0$$

$$\frac{1}{5} - \frac{1}{4} + \frac{\lambda}{3} - \frac{\lambda}{2} = 0 \quad \Leftrightarrow \quad -\frac{1}{20} - \frac{\lambda}{6} = 0$$

$$\rightarrow \frac{1}{20} + \frac{\lambda}{6} = 0 \quad \Rightarrow \frac{\lambda}{6} = -\frac{1}{20} \quad \lambda =$$

Παραγοντική Ολοκλήρωση

$$\left[f(x)\,g(x)\right]' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\int_{\alpha}^{\beta} \left[f(x)\,g(x)\right]' \,dx = \int_{\alpha}^{\beta} f'(x)\,g(x) + f(x)\,g'(x)\,dx$$

$$\Big(f(x)\,g(x)\Big)_{\alpha}^{\beta} = \int_{\alpha}^{\beta} f'(x)\,g(x)\,dx + \int_{\alpha}^{\beta} f(x)\,g'(x)\,dx$$

$$\Big(f(x)\,g(x)\Big)_{\alpha}^{\beta} - \int_{\alpha}^{\beta} f(x)\,g'(x)\,dx = \int_{\alpha}^{\beta} f'(x)\,g(x)\,dx$$

$$\boxed{\int_{\alpha}^{\beta} f'(x)\,g(x)\,dx = \Big(f(x)\,g(x)\Big)_{\alpha}^{\beta} - \int_{\alpha}^{\beta} f(x)\,g'(x)\,dx}$$

Παραγοντική ολοκλήρωση

1. $\int_0^{\pi} x\,\eta\mu x\,dx = \int_0^{\pi} x(-\sigma\upsilon\nu x)'\,dx =$

$= (-x\sigma\upsilon\nu x)_0^{\pi} - \int_0^{\pi} -\sigma\upsilon\nu x \cdot 1\,dx$

$= -(x\,\sigma\upsilon\nu x)_0^{\pi} + \int_0^{\pi} \sigma\upsilon\nu x\,dx$

$= -(\pi\sigma\upsilon\nu\pi - 0) + (\eta\mu x)_0^{\pi} =$

$= -\pi\sigma\upsilon\nu\pi + \cancel{\eta\mu\pi}^{0} - \cancel{\eta\mu 0}^{0} =$

$= \pi$

2. $\int_0^{\frac{\pi}{2}} (2x-3)\,\sigma\upsilon\nu x\,dx = \int_0^{\frac{\pi}{2}} (2x-3)\,(\eta\mu x)'\,dx$

$= \left((2x-3)\eta\mu x\right)_0^{\frac{\pi}{2}} - \int_0^{\pi/2} \eta\mu x \cdot 2\,dx =$

$= \left(\left(2\cdot\frac{\pi}{2}-3\right)\eta\mu\frac{\pi}{2} - 0\right) - 2\int_0^{\pi/2} \eta\mu x\,dx$

$= \pi - 3 - 2\,(-\sigma\upsilon\nu x)_0^{\pi/2} = \pi - 3 + 2\left(\sigma\upsilon\nu\frac{\pi}{2} - \sigma\upsilon\nu 0\right) = \pi - 5.$

3. $\int_0^1 (x^2+2x+3)\, e^x \, dx =$

$$= \int_0^1 (x^2+2x+3)\,(e^x)'\, dx =$$

$$= \left((x^2+2x+3)\, e^x\right)\Big|_0^1 - \int_0^1 e^x (2x+2)\, dx$$

$$= 6e - 3 - \int_0^1 (e^x)'(2x+2)\, dx$$

$$= 6e-3 - \left[\left[e^x(2x+2)\right]_0^1 - \int_0^1 e^x\, 2\, dx\right]$$

$$= 6e-3 - \left[4e-2-2\,(e^x)\Big|_0^1\right]$$

$$= 6e-3 - \left[4e-2-2(e-1)\right]$$

$$= 6e-3 - \left[4e-\cancel{2}-2e+\cancel{2}\right]$$

$$= 6e-3-2e = 4e-3.$$

4. $\int_1^e x^3 \ln x \, dx = \int_1^e \left(\frac{x^4}{4}\right)' \ln x \, dx =$

$= \left(\frac{x^4}{4} \ln x\right)_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{1}{x} \, dx$

$= \frac{1}{4} \left(x^4 \ln x\right)_1^e - \frac{1}{4} \int_1^e x^3 \, dx$

$= \frac{1}{4} \left(e^4\right) - \frac{1}{4} \left(\frac{x^4}{4}\right)_1^e =$

$= \frac{1}{4} e^4 - \frac{1}{16} \left(x^4\right)_1^e = \frac{1}{4} e^4 - \frac{1}{16} \left(e^4 - 1\right)$

$= \frac{1}{4} e^4 - \frac{1}{16} e^4 + \frac{1}{16}$

5. $\int_0^\pi e^x \eta\mu x \, dx$

$$I = \int_0^\pi e^x \eta\mu x \, dx$$

$$I = \int_0^\pi (e^x)' \eta\mu x \, dx$$

$$I = (e^x \eta\mu x)_0^\pi - \int_0^\pi e^x \sigma\upsilon\nu x \, dx$$

$$I = -\int_0^\pi e^x \sigma\upsilon\nu x \, dx$$

$$I = -\int_0^\pi (e^x)' \sigma\upsilon\nu x \, dx$$

$$I = -\left[(e^x \sigma\upsilon\nu x)_0^\pi - \int_0^\pi e^x (-\eta\mu x) dx \right]$$

$$I = -\left[-e^\pi - 1 + \int_0^\pi e^x \eta\mu x \, dx \right]$$

$$I = e^\pi + 1 - \int_0^\pi e^x \eta\mu x \, dx \Leftrightarrow I = e^\pi + 1 - I$$

$$2I = e^\pi + 1 \qquad I = \frac{e^\pi + 1}{2}$$

Σχόλιο

1. $\int P(x)\, e^x\, dx$

2. $\int P(x)\, \eta\mu x\, dx$

3. $\int P(x)\, \sigma\upsilon\nu x\, dx$

4. $\int P(x)\, \ln x\, dx$

5. $\underbrace{\int e^x\, \eta\mu x\, dx \quad \acute{\eta} \quad \int e^x\, \sigma\upsilon\nu x\, dx}_{\text{κυκλικό!}}$

6. $\int_1^e \ln^2 x \, dx = \int_1^e 1 \cdot \ln^2 x \, dx$

$= \int_1^e (x)' \ln^2 x \, dx =$

$= \left(x \ln^2 x\right)_1^e - \int_1^e x \cdot 2\ln x \frac{1}{x} \, dx$

$= e - 2 \int_1^e \ln x \, dx =$

$= e - 2 \int_1^e 1 \cdot \ln x \, dx =$

$= e - 2 \int_1^e (x)' \ln x \, dx =$

$= e - 2 \left[(x \ln x)_1^e - \int_1^e x \frac{1}{x} \, dx \right]$

$= e - 2 \left[e - \int_1^e 1 \, dx \right] =$

$= e - 2 \left[e - (x)_1^e \right] =$

$= e - 2 \left[\cancel{e} - \cancel{e} + 1 \right] = e - 2.$

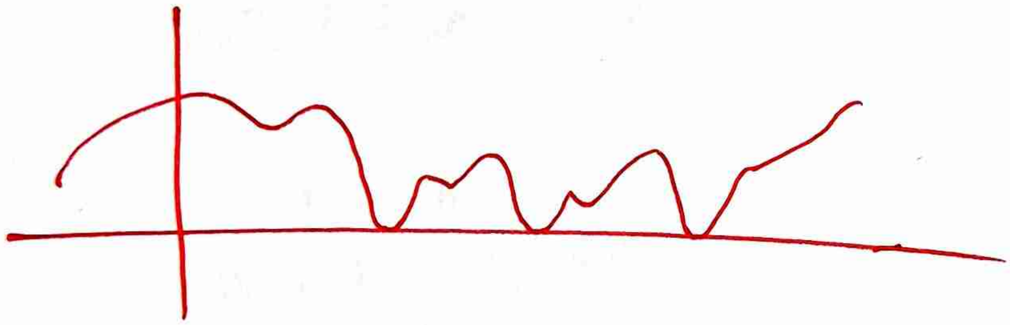
Ανισοτητες - ολοκληρωματο

1. Εστω f συνεχης στο $[\alpha, \beta]$.

Αν $f(x) \geq 0 \quad \forall x \in [\alpha, \beta]$

και $f(x)$ δεν ειναι παντου μηδεν.

Τοτε $\int_\alpha^\beta f(x)\,dx > 0$



Συμπερασμα

Αν $f(x) \geq 0$ στο $[\alpha, \beta]$

$\Longrightarrow \int_\alpha^\beta f(x)\,dx > 0$.

2. Αν $f(x) \leq g(x)$ στο $[\alpha, \beta]$

τοτε $$\int_{\alpha}^{\beta} f(x)\,dx < \int_{\alpha}^{\beta} g(x)\,dx$$

Με λιγα λογια

επιτρεπεται να

ολοκληρωνω

ανισοτητα.

2. $f(x) = \frac{e^x}{x^2+1}$

Ενότητα 37

ⓐ Μονοτονία

$$f'(x) = \frac{e^x(x^2+1) - e^x 2x}{(x^2+1)^2}$$

$$f'(x) = \frac{e^x(x^2-2x+1)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2} \geq 0$$

$f \nearrow$

ⓑ Ν.δ.ο. $2 < \int_0^2 f(x)\,dx < \frac{2e^2}{5}$.

$$0 < x < 2$$

$$f \nearrow$$

$$f(0) < f(x) < f(2)$$

$$1 < f(x) < \frac{e^2}{5}$$

$$\int_0^2 1\,dx < \int_0^2 f(x)\,dx < \int_0^2 \frac{e^2}{5}\,dx$$

$$(x)_0^2 < \int_0^2 f(x)\,dx < \frac{e^2}{5}(x)_0^2$$

$$\boxed{2 < \int_0^2 f(x)\,dx < \frac{2e^2}{5}}$$

10. $f(x) = 2e^{x-2} - x^2$

(α) $f'(x) = 2e^{x-2} - 2x$

$f''(x) = 2e^{x-2} - 2 = 2(e^{x-2} - 1)$

$\to f''(x) = 0 \Rightarrow e^{x-2} - 1 = 0$

$e^{x-2} = 1$

$x - 2 = 0 \qquad x = 2$

x		2	
f''	$-$	0	$+$
f	$\frown$		$\smile$

$y - f(2) = f'(2)(x-2)$

$y - (-2) = -2(x-2)$

$y + 2 = -2x + 4$

$\boxed{y = -2x + 2}$

(γ) i) ν.δ.ο $\int_e^{e^2} \frac{f(x)}{x}\,dx > -2e^2 + 2e + 2$

$\forall x > 2$ f κυρτή $f(x) \geq -2x + 2$

$\frac{f(x)}{x} \geq -2 + \frac{2}{x} \Rightarrow \int_e^{e^2} \frac{f(x)}{x}\,dx > \int_e^{e^2} -2 + \frac{2}{x}\,dx$

$\to \int_e^{e^2} -2 + \frac{2}{x}\,dx = \int_e^{e^2} -2\,dx + 2\int_e^{e^2} \frac{1}{x}\,dx = -2(x)_e^{e^2} + 2(\ln x)_e^{e^2}$

$= -2e^2 + 2e + 2$

ii) νδο $\int_2^3 x f(x)\,dx > -\frac{23}{3}$

Αφού αν $x>2$ η f κυρτή

$$f(x) \geq -2x+2$$

$$x f(x) \geq -2x^2+2x$$

$$\int_2^3 x f(x)\,dx > \int_2^3 -2x^2+2x\,dx$$

$\to \int_2^3 -2x^2+2x\,dx = \int_2^3 -2x^2\,dx + \int_2^3 2x\,dx$

$$= \frac{-2}{3}\left(x^3\right)_2^3 + \left(x^2\right)_2^3 =$$

$$= -\frac{2}{3}\cdot 19 + 5 = -\frac{38}{3} + \frac{15}{3} = -\frac{23}{3}$$

iii) ν.δ.ο $\int_0^1 f(e^x+2)\,dx > -2e$

$e^x+2=t \Rightarrow e^x = t-2$

$e^x\,dx = dt \Rightarrow dx = \frac{1}{e^x}\,dt \Rightarrow dx = \frac{1}{t-2}\,dt$

$$\int_3^{e+2} \frac{f(t)}{t-2}\,dt > -2e$$

$$\boxed{\int_3^{e+2} \frac{f(x)}{x-2}\,dx > -2e}$$

Αφού f κυρτή $\forall x>2$ $f(x) \geq -2x+2$

$$\frac{f(x)}{x-2} \geq \frac{-2x+2}{x-2}$$

$$\int_3^{e+2} \frac{f(x)}{x-2}\,dx > \int_3^{e+2} \frac{-2x+2}{x-2}\,dx$$

$$\int_3^{e+2} \frac{-2x+2}{x-2}\,dx = \int_3^{e+2} \frac{-2(x-2)-2}{x-2}\,dx$$

$$\begin{array}{r|l} -2x+2 & x-2 \\ \cline{2-2} -(-2x+4) & -2 \\ \hline -2 & \end{array}$$

$$= \int_3^{e+2} \frac{-2(x-2)}{x-2} - \frac{2}{x-2}\,dx$$

$$= \int_3^{e+2} -2 - \frac{2}{x-2}\,dx$$

$$= \int_3^{e+2} -2\,dx - 2\int_3^{e+2} \frac{1}{x-2}\,dx$$

$$= -2\,\left(x\right)_3^{e+2} - 2\left(\ln(x-2)\right)_3^{e+2}$$

$$= -2(e+2-3) - 2(\ln(e+2-2) - \ln 1)$$

$$= -2(e-1) - 2(\ln e - 0)$$

$$= -2e + \cancel{2} - \cancel{2} \quad \underline{= -2e}$$

11. Η f κοίλη και $f(1)=f'(1)=2$

Νδο $\displaystyle\int_0^1 \frac{f(x)}{x^2+1}\,dx < \ln 2$

$$y - f(1) = f'(1)(x-1)$$

$$y - 2 = 2(x-1)$$

$$y-2 = 2x-2$$

$$y = 2x$$

Αφού f κοίλη $f(x) \leq 2x$

$$\frac{f(x)}{x^2+1} \leq \frac{2x}{x^2+1}$$

$$\int_0^1 \frac{f(x)}{x^2+1}\,dx \leq \int_0^1 \frac{2x}{x^2+1}\,dx$$

$$\rightarrow \int_0^1 \frac{2x}{x^2+1}\,dx = \left(\ln(x^2+1)\right)_0^1 = \ln 2$$

Άρα $\displaystyle\int_0^1 \frac{f(x)}{x^2+1}\,dx < \ln 2.$

4. $f(x) = \dfrac{e^x}{x^2}, \quad x > 0$

B1 $f'(x) = \dfrac{e^x x^2 - e^x \, 2x}{x^4} = \dfrac{e^x(x^2-2x)}{x^4}$

$f'(x) = \dfrac{e^x(x-2)}{x^3}$

x	0		2	
f'		−		+
f		↘		↗

$f(x) \geq f(2)$

$f(x) \geq \dfrac{e^2}{4}$

B i) νδο $\dfrac{e^2}{2} < \int_2^4 f(x)\,dx < \dfrac{e^4}{8}$

$2 < x < 4$

$f \nearrow$

$f(2) < f(x) < f(4)$

$\dfrac{e^2}{4} < f(x) < \dfrac{e^4}{16}$

$\int_2^4 \dfrac{e^2}{4}\,dx < \int_2^4 f(x)\,dx < \int_2^4 \dfrac{e^4}{16}\,dx$

$\dfrac{e^2}{4} \cdot (x)_2^4 < \int_2^4 f(x)\,dx < \dfrac{e^4}{16}\,(x)_2^4$

$\boxed{\dfrac{e^2}{2} < \int_2^4 f(x)\,dx < \dfrac{e^4}{8}}.$

11) NSo $\int_1^5 f(x)\,dx > e^2$

$$f(x) \geq \frac{e^2}{4}$$

$$\int_1^5 f(x)\,dx > \int_1^5 \frac{e^2}{4}\,dx$$

$$\int_1^e f(x)\,dx > \frac{e^2}{4}(x)_1^5$$

$$\int_1^e f(x)\,dx > e^2$$

Επομενο Μαθημα

35

2

3

4

5

6

15 α β γ

16 α β δ

17 α

18 α β δ στ

37

3

5

6

7

8