Θεμα 150

$$f(x) = \text{εφ}x - x + \ln(\text{συν}x), \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(α) Μονοτονία

$$f'(x) = \frac{1}{\text{συν}^2 x} - 1 + \frac{-\text{ημ}x}{\text{συν}x} = \frac{1}{\text{συν}^2 x} - 1 - \frac{\text{ημ}x}{\text{συν}x}$$

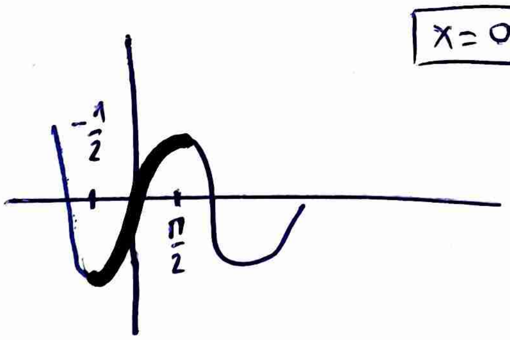
$$f'(x) = \frac{1 - \text{συν}^2 x - \text{ημ}x\,\text{συν}x}{\text{συν}^2 x} = \frac{\text{ημ}^2 x - \text{ημ}x\,\text{συν}x}{\text{συν}^2 x}$$

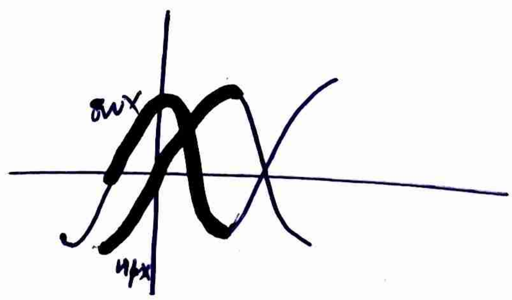
$$\boxed{f'(x) = \frac{\text{ημ}x\,(\text{ημ}x - \text{συν}x)}{\text{συν}^2 x}}$$

$$f'(x) = 0 \rightarrow \text{ημ}x\,(\text{ημ}x - \text{συν}x) = 0$$

$\text{ημ}x = 0$ ή $\text{ημ}x = \text{συν}x$

$\boxed{x = 0}$ $\quad$ $x = \frac{\pi}{4}$





x	$-\frac{\pi}{2}$		0		$\frac{\pi}{4}$		$\frac{\pi}{2}$
$ημx$		$-$	0	$+$		$+$	
$ημx - συνx$		$-$		$-$	0	$+$	
f'		$+$		$-$		$+$	
f		$\nearrow$		$\searrow$		$\nearrow$	

(B) $$f'(x) = \frac{1}{συν^2x} - 1 - \frac{ημx}{συνx}$$

$$f''(x) = \frac{+2συνx\,ημx}{συν^4x} - \frac{1}{συν^2x}$$

$$f''(x) = \frac{2ημx}{συν^3x} - \frac{συνx}{συν^3x} = \frac{2ημx - συνx}{συν^3x}$$

$$\boxed{f''(x) = \frac{2ημx - συνx}{συν^3x}}$$

$$φ(x) = 2ημx - συνx$$

Αρκει νδο η φ(x) εχει μοναδικη ριζα

αρχικα!

$\varphi\left(-\frac{\pi}{2}\right) = 2\,\eta\mu\left(-\frac{\pi}{2}\right) - \sigma\upsilon\nu\left(-\frac{\pi}{2}\right) =$

$= -2\,\eta\mu\frac{\pi}{2} - \cancel{\sigma\upsilon\nu\frac{\pi}{2}}$

$= -2$

$\eta\mu(-x) = -\eta\mu x$

$\sigma\upsilon\nu(-x) = \sigma\upsilon\nu x$

$\varphi\left(\frac{\pi}{2}\right) = 2\,\eta\mu\frac{\pi}{2} - \sigma\upsilon\nu\frac{\pi}{2}$

$\varphi\left(\frac{\pi}{2}\right) = 2 - 0 = 2$

$\varphi\left(-\frac{\pi}{2}\right)\varphi\left(\frac{\pi}{2}\right) < 0$ Bolzano $\exists x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

τ.ω $\varphi(x_0) = 0$ δηλαδή $f''(x_0) = 0$.

x	$-\pi/2$	x_0	$\pi/2$
f''	$-$	0	$+$
f	$\cap$		$\cup$

$\varphi\left(\frac{\pi}{2}\right) = 2$

Θέλω νδο $x_0 < \frac{\pi}{6}$

Γνωρίζω ότι $\varphi(x_0)=0 \Rightarrow 2\eta\mu x_0 - \sigma\upsilon\nu x_0 = 0$

$\boxed{2\eta\mu x_0 = \sigma\upsilon\nu x_0}$

Έστω $x_0 \geq \frac{\pi}{6}$

Αφού $f''(x)>0 \ \forall x > x_0$

τοτε $f' \nearrow$

$f'(x_0) \geq f'\left(\frac{\pi}{6}\right)$

$$\frac{\eta\mu x_0(\eta\mu x_0 - \sigma\upsilon\nu x_0)}{\sigma\upsilon\nu^2 x_0} \geq \frac{\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)}{\frac{3}{4}}$$

$$\frac{\cancel{\eta\mu x_0}(\eta\mu x_0 - \sigma\upsilon\nu x_0)}{4\eta\mu^2 x_0} \geq \frac{\frac{1}{4}-\frac{\sqrt{3}}{4}}{\frac{3}{4}}$$

$$\frac{\eta\mu x_0 - 2\eta\mu x_0}{4\eta\mu x_0} \geq \frac{1-\sqrt{3}}{3}$$

$$\frac{-\cancel{\eta\mu x_0}}{4\cancel{\eta\mu x_0}} \geq \frac{1-\sqrt{3}}{3}$$

$$-\frac{1}{4} \geq \frac{1-\sqrt{3}}{3} \Rightarrow -3 \geq 4-4\sqrt{3}$$

$4\sqrt{3} \geq 7$

$48 \geq 49$

Άτοπο!

$\frac{\pi}{2} > x_0 > \frac{\pi}{6}$

$\eta\mu x \nearrow$

$\eta\mu\frac{\pi}{2} > \eta\mu x_0 > \eta\mu\frac{\pi}{6} \Rightarrow 1 > \eta\mu x_0 > \frac{1}{2}$

(γ) Νδο η εξισωση $f(x) - f(x_0) = (x - x_0) f'(x_0)$

εχει μοναδικη λυση

Προφανης ριζα $x = x_0$

Η εφαπτομενη στο x_0 ειναι:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$\boxed{y = f'(x_0)(x - x_0) + f(x_0)}$$

Οταν $x > x_0$ τοτε η f κυρτη.

$$f(x) > f'(x_0)(x - x_0) + f(x_0)$$

Οταν $x < x_0$ τοτε η f κοιλη

$$f(x) < f'(x_0)(x - x_0) + f(x_0)$$

(δ) Εξίσωση $f\left(\frac{\pi}{4}-x\right)=0$ στο $\left(0,\frac{\pi}{2}\right)$.

$0<x<\frac{\pi}{2}$

$0>-x>-\frac{\pi}{2}$

$\frac{\pi}{4}>\frac{\pi}{4}-x>\frac{\pi}{4}-\frac{\pi}{2}$

$\frac{\pi}{4}>\frac{\pi}{4}-x>-\frac{\pi}{4}$

$\forall x<\frac{\pi}{4}$

$f(x)\le f(0)$

$f(x)\le 0.$

Μονο το $f(0)=0$

$f\left(\frac{\pi}{4}-x\right)=0$

$\frac{\pi}{4}-x=0$

$x=\frac{\pi}{4}$

Θεμα ΙΙΙ

$f(0)=1$

$f(x)\,f'(x) = x \quad \forall x \in \mathbb{R}$ $\quad$ f παρ/μη.

ⓐ Ν.δ.ο $f(x)=\sqrt{x^2+1}$

$2f(x)f'(x) = 2x$

$\left(f^2(x)\right)' = (x^2)'$

$f^2(x) = x^2 + c \longrightarrow f^2(x) = x^2+1.$

$\underline{x=0}$

$f^2(0) = 0 + c$

$1 = c.$

$f^2(x) = \sqrt{x^2+1}^{\,2}$

$|f(x)| = |\sqrt{x^2+1}|$

$|f(x)| = \sqrt{x^2+1}$

Ριζα f(x)

$f(x) = 0$

$|f(x)| = 0$

$\sqrt{x^2+1} = 0$

Ατοπο

$f(x) \neq 0$

$f(x)>0$ ή $f(x)<0$

$f(0)=1$

$f(x)>0$

$f(x) = \sqrt{x^2+1}$

(B) . Ν.δ.ο η $y=x$ ασυμπτωτη στο $+\infty$.

$$\lim_{x\to+\infty}(f(x)-x) = \lim_{x\to+\infty} \sqrt{x^2+1} - x$$

$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to+\infty} \frac{\cancel{\sqrt{x^2+1}}^{2} - x^2}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}+x}$$

$$= 0$$ ✓

Ν.δ.ο $f(x) > x$

$$\sqrt{x^2+1} > x$$

Αν $x \geq 0$ τοτε $\sqrt{x^2+1}^{\,2} > x^2$

$$\cancel{x^2}+1 > \cancel{x^2}$$

$$1 > 0$$ ✓

Αν $x<0$ ισχυει παλι,

Ισχυει παντα.

(γ) $f(x) = \sqrt{x^2+1}$

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

x		0	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(0)$

$f(x) \geq 1$

$$f''(x) = \frac{\sqrt{x^2+1} - x \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x^2+1}$$

$$f''(x) = \frac{x^2+1 - x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+1)\sqrt{x^2+1}} > 0$$

f κυρτή.

(δ) Νδο $\int_1^2 f(x) \ln x \, dx > 2\ln 2 - \frac{3}{4}$

$f(x) \geq 1$

$f(x) \ln x \geq \ln x$

$$\int_1^2 f(x) \ln x \, dx > \int_1^2 \ln x \, dx$$

$$\rightarrow \int_1^2 \ln x \, dx = (x\ln x)_1^2 - \int_1^2 x \cdot \frac{1}{x} dx =$$

$$= 2\ln 2 - (x)_1^2 = 2\ln 2 - 1$$

Έπεται ότι $\int_1^2 f(x)\ln x\,dx > 2\ln 2 - 1$

$\frac{3}{4} < 1$

$-\frac{3}{4} > -1$

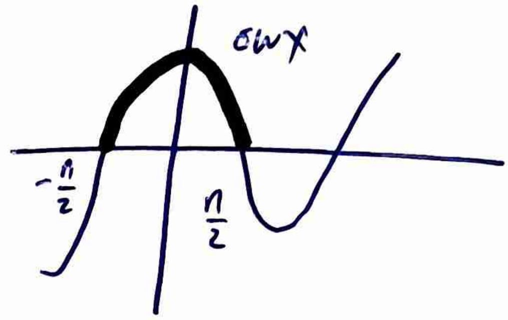
$2\ln 2 - \frac{3}{4} > 2\ln 2 - 1$

Άρα $\int_1^2 f(x)\ln x\,dx > 2\ln 2 - \frac{3}{4}$

Θεμα 109

$$f(x) = εφx + ημx \qquad x \in \left(-\frac{π}{2}, \frac{π}{2}\right)$$

α) $f'(x) = \dfrac{1}{συν^2 x} + συνx > 0$



$f \uparrow \qquad D_{f^{-1}} = ΣT_f$

$$\lim_{x\to -\frac{π}{2}^+} f(x) = \lim_{x\to -\frac{π}{2}} \left(\frac{ημx}{συνx} + ημx\right) = \frac{-1}{0} - 1 = -\infty$$

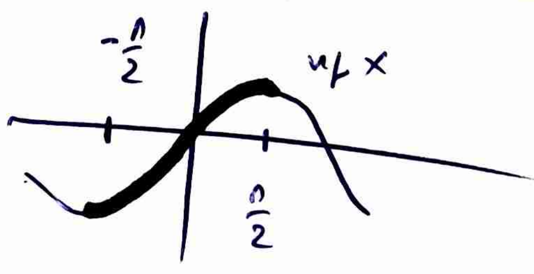
$$\lim_{x\to \frac{π}{2}^-} f(x) = \lim_{x\to \frac{π}{2}^-} \left(\frac{ημx}{συνx} + ημx\right) = \frac{1}{0} + 1 = +\infty$$

$$ΣT_f = D_{f^{-1}} = \mathbb{R}.$$

β). $f''(x) = \dfrac{+2συνx\,ημx}{συν^4 x} - ημx$

$$f''(x) = \frac{2ημx}{συν^3x} - \frac{ημx\,συν^3x}{συν^3x} = \frac{ημx\overbrace{(2-συν^3x)}^{+}}{\underbrace{συν^3x}_{+}}$$

x	$-π/2$ 0	$π/2$
f''	$-$	$+$
f	∩	∪ .



$y - f(0) = f'(0)(x-0)$

$y - 0 = 2x$

$\boxed{y = 2x}$

(γ) $2f^{-1}(x) > x$ Ανίσωση.

$f^{-1}(x) > \frac{x}{2}$

$f \uparrow$

$f(f^{-1}(x)) > f(\frac{x}{2})$

$x > f(\frac{x}{2})$ $\Rightarrow 2t > f(t)$

$\frac{x}{2} = t \Rightarrow x = 2t$

$f(t) < 2t$

$f(x) < 2x$

$x < 0$

$\forall x < 0$

f κοίλη

$f(x) < 2x$

(δ). $\lim_{x \to +\infty} \dfrac{f^{-1}(x)}{x} \underset{x=f(t)}{\overset{f^{-1}(x)=t}{=\!=\!=\!=}}$

$= \lim_{t \to \frac{\pi}{2}} \dfrac{t}{f(t)} = 0.$

αφού $\lim_{x \to \frac{\pi}{2}} f(x) = +\infty.$

Θεμα 149

$$f(x) = e^{x-1} - \frac{1}{x} \quad , \; x \neq 0$$

(α) $f'(x) = e^{x-1} + \frac{1}{x^2} > 0$ $\quad f \nearrow$ στο $(-\infty,0)$ και $(0,+\infty)$

(β) $f''(x) = e^{x-1} + \dfrac{-2x}{x^4} = e^{x-1} - \dfrac{2}{x^3}$

$$f''(x) = \frac{x^3 e^{x-1} - 2}{x^3}$$

$\varphi(x) = x^3 e^{x-1} - 2$

$\varphi'(x) = 3x^2 e^{x-1} + x^3 e^{x-1} = e^{x-1} x^2 (3+x)$

x		-3	
φ'	$-$		$+$
φ	$-2 \searrow$	α ⊝	$\nearrow$

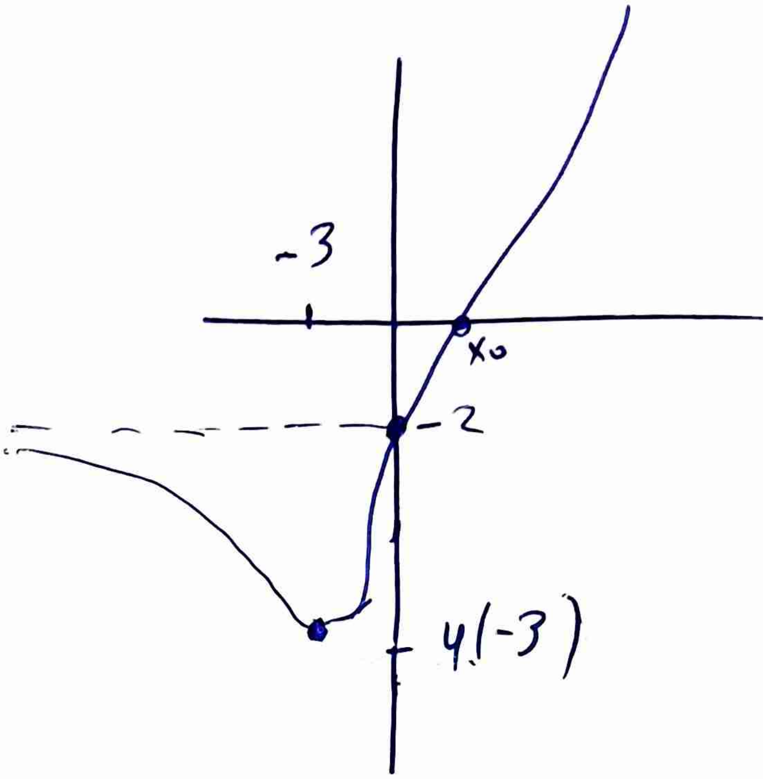
$\varphi(x) \geq \varphi(-3)$

$\varphi(x) \geq -27e^{-4} - 2$

$$\lim_{x \to -\infty} \varphi(x) = \lim_{x \to -\infty} x^3 e^{x-1} - 2 = -2$$

$$\to \lim_{x \to -\infty} \frac{x^3}{e^{1-x}} = \lim_{x \to -\infty} \frac{3x^2}{-e^{1-x}} = \lim_{x \to -\infty} \frac{6x}{e^{1-x}} = 0$$

$\lim_{x \to +\infty} \varphi(x) = +\infty$



$x < -3$

- φ συνεχής
- $\varphi \downarrow$
- $\Sigma T_\varphi = (\varphi(-3), -2)$

$\downarrow$

αρνητικό

Το $0 \notin \Sigma T_\varphi$

$x > -3$

- φ συνεχής
- $\varphi \nearrow$
- $\Sigma T_\varphi = [\varphi(-3), +\infty)$

Το $0 \in \Sigma T_\varphi$ άρα

$\exists!\, x_0$ τ.ω $\varphi(x_0) = 0$

$f''(x_0) = 0$

x		0		x_0	
$\varphi(x)$	$-$		$-$		$+$
x^3	$-$		$+$		$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

(γ) $\int_1^2 e^{f(x)} f(x) f'(x)\, dx = \int_0^{e-\frac{1}{2}} e^t t \, dt$

$f(x) = t$

$f'(x)\, dx = dt$

$= \left(te^t\right)_0^{e-\frac{1}{2}} - \int_0^{e-\frac{1}{2}} e^t \, dt$

$= \left(e-\frac{1}{2}\right) e^{e-\frac{1}{2}} - \left(e^t\right)_0^{e-\frac{1}{2}}$

(δ) $g: (0,+\infty)$ $\qquad g'(x) < 0 \qquad g(1)=0$

$$x\,g(x) + 1 = x\,e^{x-1}$$

$$g(x) + \frac{1}{x} = e^{x-1}$$

$$g(x) = e^{x-1} - \frac{1}{x}$$

$$g(x) = f(x)$$

Προφανής ρίζα $x = 1$

$$\underbrace{g(x) - f(x)}_{h(x)} = 0$$

$$h'(x) = g'(x) - f'(x) < 0$$

$h \downarrow$ το $x = 1$ μοναδική

Θεμα 107

$$f(x) = \ln x + x + e^{-x} \quad , x > 0$$

(α) $f'(x) = \frac{1}{x} + 1 - e^{-x}$

$$f'(x) = \frac{1}{x} + 1 - \frac{1}{e^x} = \frac{e^x + xe^x - x}{xe^x}$$

$$\boxed{f'(x) = \frac{e^x(x+1) - x}{xe^x}}$$

$$\varphi(x) = e^x(x+1) - x$$

$$\varphi'(x) = e^x(x+1) + e^x - 1$$

$$\varphi''(x) = e^x(x+1) + e^x + e^x = e^x(x+1+1+1)$$

$$\varphi''(x) = e^x(x+3) > 0$$

x	0
φ''	$+$
φ'	$\nearrow$ $+$
φ	$\nearrow$ $+$

$x > 0 \Rightarrow \varphi'(x) > \varphi'(0)$

$\varphi'(x) > 1$

$x > 0 \rightarrow \varphi(x) > \varphi(0)$

$\varphi(x) > 1$

$f'(x) > 0 \quad f \nearrow$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\ln x + x + e^{-x} \right) = -\infty$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \left(\ln x + x + e^{-x} \right) = +\infty$$

$\Sigma T_f = \mathbb{R}$ αρα $D_{f^{-1}} = \mathbb{R}$.

Ⓑ. $\ln f^{-1}(x) + f^{-1}(x) = x^2 - e^{-f^{-1}(x)}$

$$\underbrace{\ln f^{-1}(x) + f^{-1}(x) + e^{-f^{-1}(x)}}_{f(f^{-1}(x))} = x^2$$

$$f(f^{-1}(x)) = x^2$$

$$x = x^2$$

$$x^2 - x = 0$$

$x = 0$ $x = 1$

Ⓓ ΝΔΟ $\exists x_0 < 1$ τ.ω το $\lim_{x\to x_0} \frac{1}{f(x)}$ να

μην υπαρχει.

- f συνεχη
- $f \nearrow$
- $\Sigma T_f = \mathbb{R}$

$\Big\}$ $\exists!\, x_0$ τ.ω $f(x_0)=0.$

x		x_0	
$f(x)$	$\nearrow$ $-$	0	$+$ $\nearrow$

$$\lim_{x\to x_0^-} \frac{1}{f(x)} = -\infty$$

$$\lim_{x\to x_0^+} \frac{1}{f(x)} = +\infty$$

$\Big\}$ Το όριο δεν υπάρχει

Έστω ότι $x_0 \geq 1$

$f \nearrow$

$f(x_0) \geq f(1)$

$0 \geq 1 + \frac{1}{e}$ Άτοπο

$x_0 < 1.$

(δ) Ν.δ.ο. $e^{-x} - f(\varepsilon\varphi x) < \ln\frac{1}{x} - x$

$\forall x \in (0, \frac{\pi}{2})$,

$$e^{-x} - \ln\frac{1}{x} + x < f(\varepsilon\varphi x)$$

$$e^{-x} - (\ln 1 - \ln x) + x < f(\varepsilon\varphi x)$$

$$e^{-x} + \ln x + x < f(\varepsilon\varphi x)$$

$$f(x) < f(\varepsilon\varphi x)$$

$$\xLeftrightarrow{f \nearrow}$$

$$x < \varepsilon\varphi x, \quad \forall x \in (0, \tfrac{\pi}{2})$$

$$\underbrace{x - \varepsilon\varphi x}_{g(x)} < 0$$

$$g'(x) = 1 - \frac{1}{\sigma\upsilon\nu^2 x} = \frac{\sigma\upsilon\nu^2 x - 1}{\sigma\upsilon\nu^2 x} = \frac{-\eta\mu^2 x}{\sigma\upsilon\nu^2 x}$$

$$g'(x) = -\varepsilon\varphi^2 x < 0 \qquad g \downarrow.$$

$$0 < x < \frac{\pi}{2} \xrightarrow{g\downarrow} g(0) > g(x) > g(\tfrac{\pi}{2})$$

$$\underline{\underline{0 > g(x)}}$$

Επόμενο Μάθημα

Τα πρώτα

30

θέματα

Μπαρόλα.

Η 1η ώρα του μαθήματος

Εύρεση τύπου με

αντιπαραγώγιση.