14. $y = x^3 + x$, $\underline{x < 0}$ $\qquad$ $\underline{y'(t) = 4x'(t)}$

$y(t) = x^3(t) + x(t)$

$y'(t) = 3x^2(t)\, x'(t) + x'(t)$

$4\cancel{x'(t)} = 3x^2(t)\, \cancel{x'(t)} + \cancel{x'(t)}$ $\qquad$ $\underline{x'(t) > 0}$

$4 = 3x^2(t) + 1$

$3 = 3x^2(t)$

$x^2(t) = 1$

$x(t) = 1$ ή $x(t) = -1$

$A(-1,\ -2,\ )$

Ενότητα

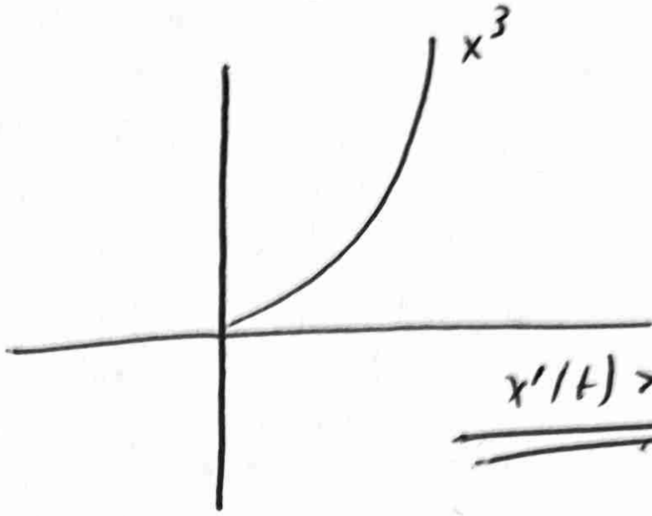
19

13. $y'(t) = x'(t)$

$y = x^3$

$y(t) = x^3(t)$

$y'(t) = 3x^2(t)\, x'(t)$



$x'(t) > 0$

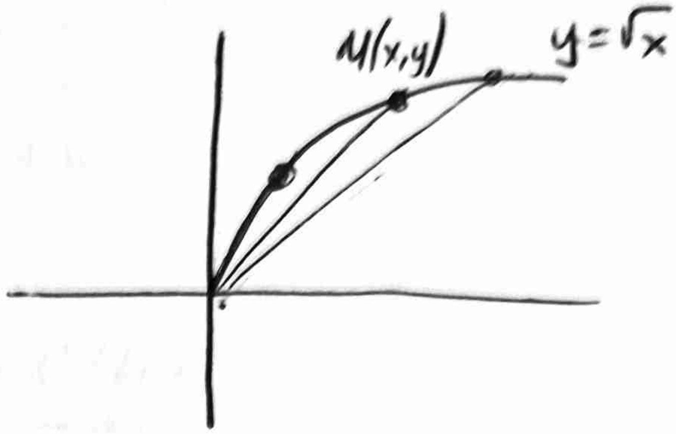
$\cancel{x'(t)} = 3x^2(t)\, \cancel{x'(t)}$

$1 = 3x^2(t)$

$\frac{1}{3} = x^2(t)$

$x(t) = \frac{\sqrt{3}}{3}$ ή $\cancel{x(t) = -\frac{\sqrt{3}}{3}}$

17.



$$OM = \sqrt{(x-0)^2 + (y-0)^2}$$

$$OM = \sqrt{x^2 + y^2}$$

$$OM = \sqrt{x^2 + \sqrt{x}^2}$$

$$OM = \sqrt{x^2 + x}$$

$$d'(t) = 3$$

$$d = \sqrt{x^2 + x}$$

Οταν $x = 4$ τοτε $x(t_1) = 4$

$$d(t) = \sqrt{x^2(t) + x(t)}$$

$$d'(t) = \frac{2x(t)x'(t) + x'(t)}{2\sqrt{x^2(t) + x(t)}}$$

$\underline{t = t_1}$

$$d'(t_1) = \frac{2x(t_1)x'(t_1) + x'(t_1)}{2\sqrt{x^2(t_1) + x(t_1)}}$$

$$3 = \frac{2 \cdot 4\, x'(t_1) + x'(t_1)}{2\sqrt{4^2+4}}$$

$$3 = \frac{8\, x'(t_1) + x'(t_1)}{2\sqrt{20}}$$

$$6\sqrt{20} = 9\, x'(t_1)$$

$$x'(t_1) = \frac{6\sqrt{4}\sqrt{5}}{9}$$

$$x'(t_1) = \frac{12\sqrt{5}}{9}$$

$$x'(t_1) = \frac{4\sqrt{5}}{3}$$

18.

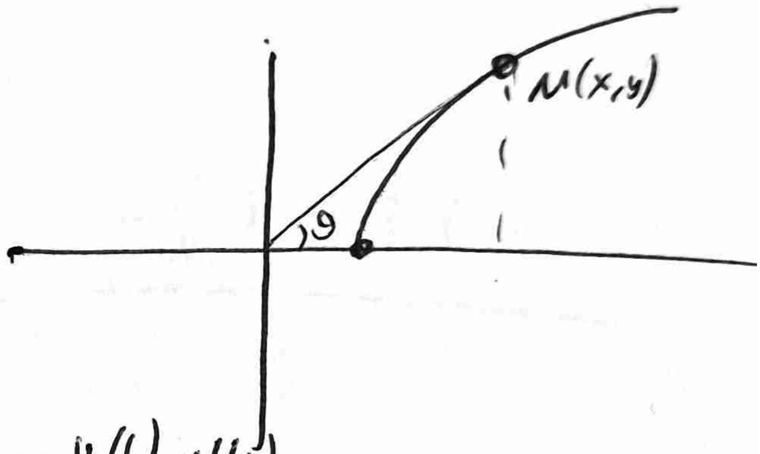
$y = \sqrt{x-1}$

Τη χρονική στιγμή t_1 το μυρμήγκι

περνάει από $M(5,2)$

$\Rightarrow$ $\boxed{x(t_1) = 5 \qquad y(t_1) = 2}$

$x'(t_1) = -3$



$$\varepsilon\varphi\,\hat{\theta} = \frac{y}{x}$$

$$\varepsilon\varphi\,\theta(t) = \frac{y(t)}{x(t)}$$

$$\frac{1}{\sigma\upsilon\nu^2\theta(t)}\,\theta'(t) = \frac{y'(t)\,x(t) - y(t)\,x'(t)}{x^2(t)}$$

$$\frac{1}{\frac{1}{1+\varepsilon\varphi^2\theta(t)}}\,\theta'(t) = \frac{y'(t)\,x(t) - y(t)\,x'(t)}{x^2(t)},$$

$$\left(1+\frac{y^2(t)}{x^2(t)}\right)\theta'(t)=\frac{y'(t)\,x(t)-y(t)\,x'(t)}{x^2(t)}$$

$$\underline{t=t_1}$$

$$\left(1+\frac{y^2(t_1)}{x^2(t_1)}\right)\theta'(t_1)=\frac{y'(t_1)x(t_1)-y(t_1)x'(t_1)}{x^2(t_1)}.$$

$$\left(1+\frac{4}{25}\right)\theta'(t_1)=\frac{-\frac{3}{4}5-2\cdot(-3)}{9}$$

$$\frac{29}{25}\theta'(t_1)=\frac{\frac{-15}{4}+6}{9}\Rightarrow\frac{29}{25}\theta'(t_1)=\frac{9}{36}$$

$$\theta'(t_1)=\frac{9\cdot 25}{36\cdot 29}$$

$$\theta'(t_1)=\frac{25}{4\cdot 29}=$$

$$=\frac{25}{4\cdot 29}=\frac{25}{116}$$

$$y(t)=\sqrt{x(t)-1}$$

$$y'(t)=\frac{x'(t)}{2\sqrt{x(t)-1}}$$

$$y'(t_1)=\frac{-3}{2\sqrt{4}}=\frac{-3}{4}$$

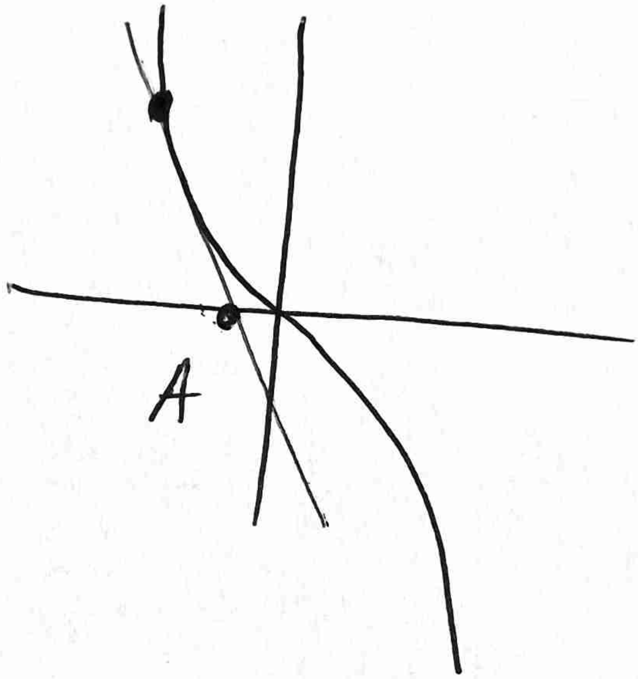
21. $f(x) = -x^3 \quad , x \leq 0$

(α) $y - f(a) = f'(a)(x-a)$

$y + a^3 = -3a^2(x-a)$ $\qquad a'(t) = -2a(t)$

$y = -3a^2x + 3a^3 - a^3$

$$\boxed{y = -3a^2x + 2a^3}$$



Για $y=0$

$0 = -3a^2x + 2a^3$

$3a^2x = 2a^3$

$x = \frac{2a}{3} \longrightarrow$ Η τετμημένη του A

$x(t) = \frac{2}{3}a(t)$

$x'(t) = \frac{2}{3}a'(t) = \frac{2}{3}(-2a(t)) = -\frac{4}{3}a(t)$

$\xrightarrow{t=t_1}$

$x'(t_1) = -\frac{4}{3}a(t_1) = -\frac{4}{3}\cdot(-6) = \boxed{8}$

22. $f(x) = \ln x$ $\qquad$ $\alpha'(t) = 2\,a(t)$

Έστω t_1 η χρονική στιγμή που το κινητό έχει τετμημένη e

$$\boxed{\begin{aligned} x(t_1) &= e \\ y(t_1) &= 1 \end{aligned}}$$

$M(a, f(a))$

$y - f(a) = f'(a)(x - a)$

$y - \ln a = \frac{1}{a}(x - a)$

$\underline{y = 0}$

$-\ln a = \frac{1}{a}(x - a)$

$-a \ln a = x - a$

$$\boxed{x = a - a\ln a}$$ $\longrightarrow$ η τετμημένη του σημείου τομής της εφαπτομένης με τον $x'x$

$x(t) = a(t) - a(t)\ln a(t)$

$y = \ln x$

$y(t) = \ln x(t)$

$y'(t) = \frac{x'(t)}{x(t)}$

$\underline{t = t_1}$

$y'(t_1) = \frac{x'(t_1)}{x(t_1)}$

$y'(t_1) = \frac{2\cancel{x(t_1)}}{\cancel{x(t_1)}}$

$y'(t_1) = 2$

$$x'(t) = a'(t) - \left(a'(t) \ln a(t) + a(t) \frac{1}{a(t)} a'(t) \right)$$

$$x'(t) = \cancel{a'(t)} - a'(t) \ln a(t) - \cancel{a'(t)}$$

$$x'(t) = -a'(t) \ln a(t)$$

$$x'(t) = -2a(t) \ln a(t)$$

$$\underline{t = t_1}$$

$$x'(t_1) = -2a(t_1) \ln a(t_1)$$

$$x'(t_1) = -2e \ln e$$

$$\underline{x'(t_1) = -2e}$$

$$\varepsilon\varphi\,\theta = f'(\alpha)$$

$$\varepsilon\varphi\,\theta(t) = \frac{1}{a(t)}$$

$$\frac{1}{\sigma\upsilon\nu^2\theta(t)}\,\theta'(t) = \frac{-1}{a^2(t)}\,a'(t)$$

$$\theta'(t) = \frac{-a'(t)}{a^2(t)}\,\frac{1}{1+\varepsilon\varphi^2\theta(t)}$$

$$\theta'(t) = -\frac{2a(t)}{a^2(t)}\cdot\frac{1}{1+\frac{1}{a^2(t)}}$$

$$\theta'(t) = -\frac{2}{\cancel{a(t)}}\cdot\frac{a^{\cancel{2}}(t)}{a^2(t)+1}$$

$$\theta'(t_1) = +\,\frac{-2e}{e^2+1}.$$

Ενοτητα 31

Ασυμπτωτες

Προκυπται για ευθειες.

Υπαρχουν 3 ειδη

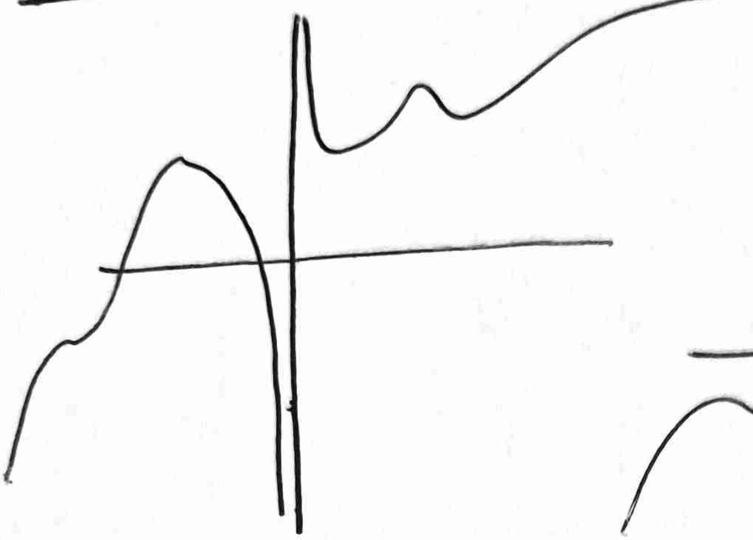
1. κατακορυφες

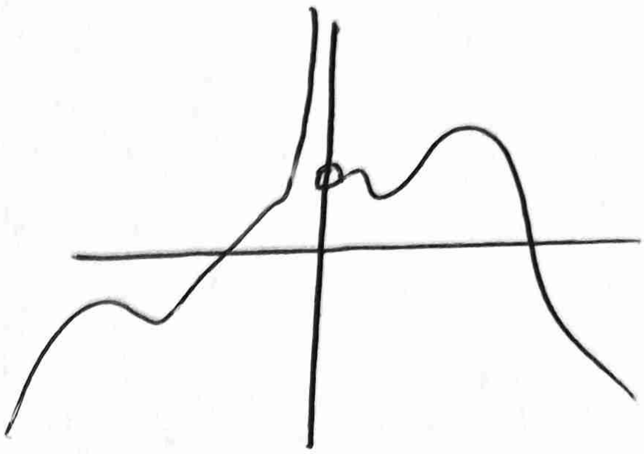
2. οριζοντιες

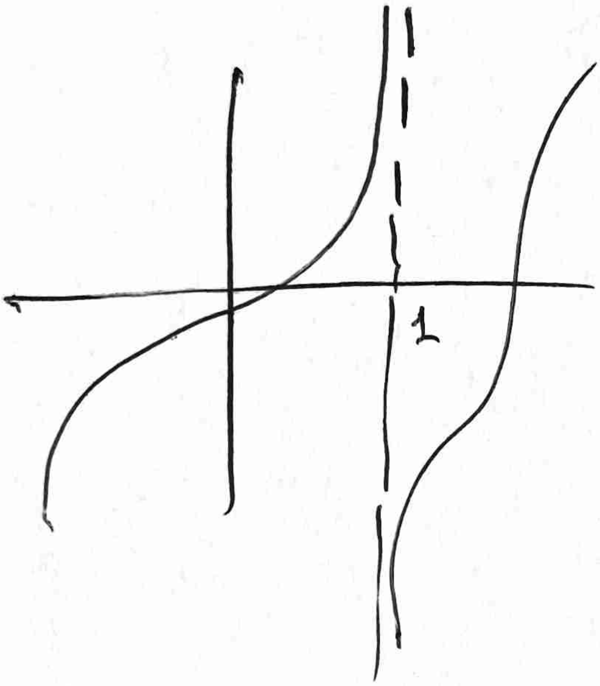
3. πλαγιες.

κατακορυφη

$\varepsilon_1 : x=0$ κατακορυφη







$\varepsilon : x=1$

Εαν τουλαχιστον ενα απο τα ορια

$$\left.\begin{array}{l} \lim\limits_{x\to x_0^-} f(x) = \pm\infty \\ \quad\quad\quad ή \\ \lim\limits_{x\to x_0^+} f(x) = \infty \end{array}\right\} \varepsilon : x = 0$$

κατακορυφη.

3. (β) $f(x) = \dfrac{x}{x-1}$ $\quad D_f = \mathbb{R} - \{1\}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{x-1} = -\infty$$

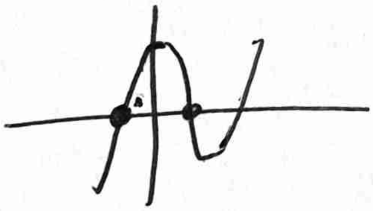
$$\boxed{\varepsilon : x = 1}$$

κατακόρυφη

(δ) $f(x) = \varepsilon\varphi x - x \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\lim_{x\to -\frac{\pi}{2}^+} f(x) = \lim_{x\to -\frac{\pi}{2}^+} \frac{\eta\mu x}{\sigma\upsilon\nu x} - \left(-\frac{\pi}{2}\right) = (-1)(+\infty) + \frac{\pi}{2}$$

$$= -\infty$$



$$\boxed{\varepsilon_1 : x = -\frac{\pi}{2}}$$

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^-} \eta\mu x \frac{1}{\sigma\upsilon\nu x} - \frac{\pi}{2} =$$

$$= 1(+\infty) - \frac{\pi}{2} = +\infty.$$

$$\varepsilon_2 : x = \frac{\pi}{2}$$

(7) $f(x) = \ln \frac{x}{1-x}$ $\qquad D_f = (0,1)$

πρέπει $\frac{x}{1-x} > 0$ και $1-x \neq 0$

$x \neq 1$

x		0		1	
x	$-$		$+$		$+$
$1-x$	$+$		$+$	0	$-$
$\frac{}{}$	$-$		$+$		$-$

$x \in (0,1)$

$$\lim_{x \to 0^+} \ln\left(\frac{x}{1-x}\right) = -\infty$$

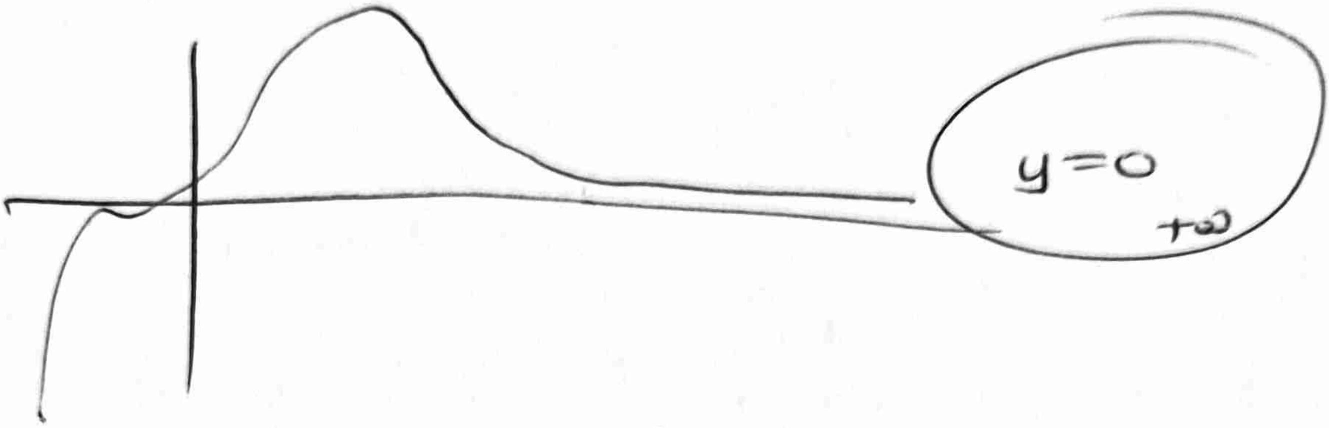
Ε.Α. $x = 0$

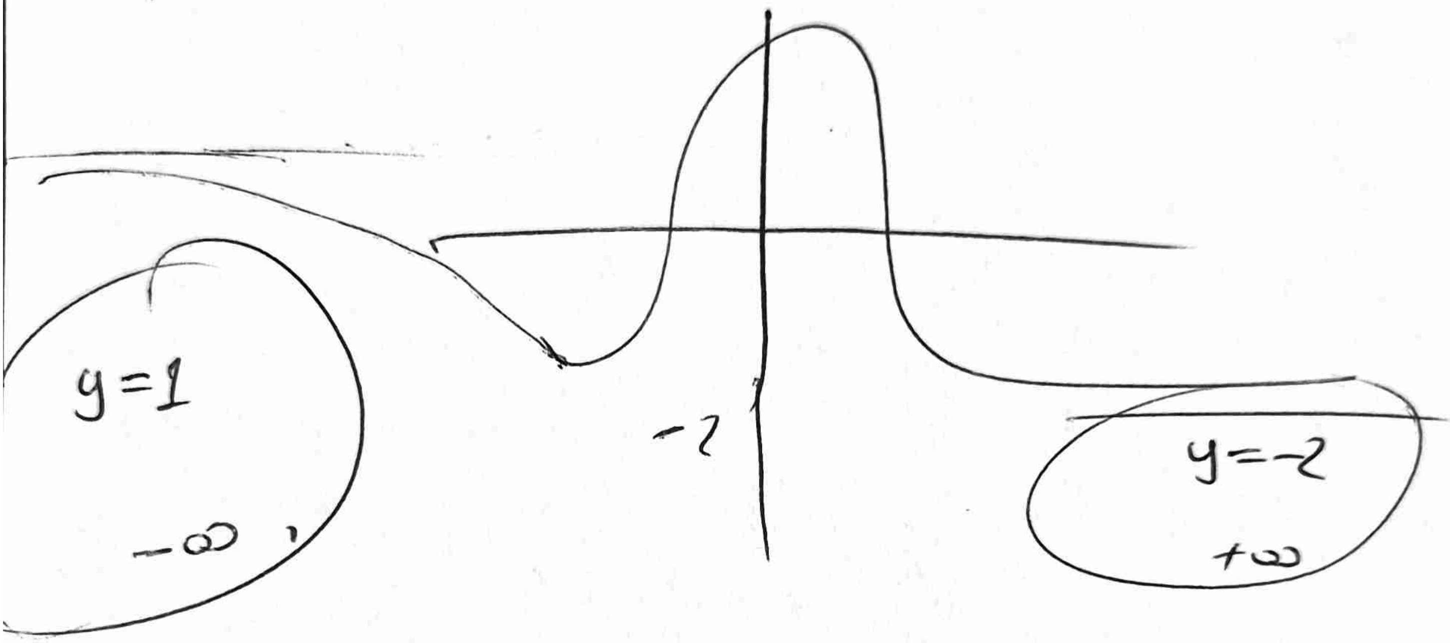
$$\lim_{x \to 1^-} \ln\left(\frac{x}{1-x}\right) = \ln +\infty = +\infty$$

Ε.Α. $x = 1$

x		1	
$1-x$	$+$		$-$

Οριζοντια





Αν $\lim_{x \to \infty} f(x) = \ell \in \mathbb{R}$

τοτε $y = \ell$ οριζοντια

στο ∞.

4. (B) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ $\quad D_f = \mathbb{R}$

Αφου $D_f = \mathbb{R}$ δεν εχει κατακορυφη.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$$

$$= \lim_{x \to -\infty} \frac{\cancel{x}}{-\cancel{x}\sqrt{1+\frac{1}{x^2}}} = \lim_{x \to -\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}}$$

$$= -1$$

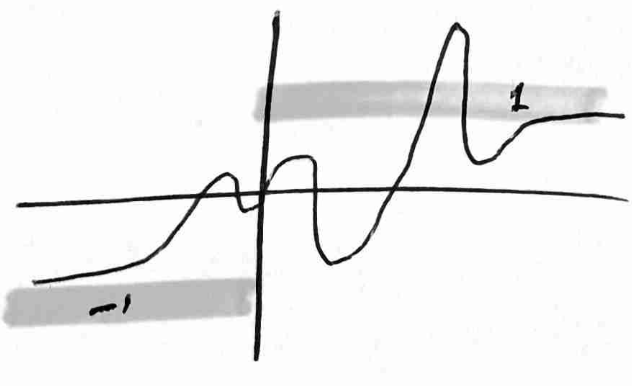
$\varepsilon: y = -1$

$-\infty$

$$\lim_{x \to +\infty} f(x) = 1$$

$\varepsilon: y = 1$

$+\infty$



(ε) $f(x) = x\,\eta\mu\frac{1}{x}$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\,\eta\mu\frac{1}{x} = \lim_{x\to+\infty} \cancel{x}\,\frac{\eta\mu\frac{1}{x}}{\frac{1}{x}}\,\cancel{\frac{1}{x}} = 1$$

$\varepsilon \colon y = 1$

$\pm\infty$

(8) $f(x) = e^{-x^2}, \; x \in \mathbb{R}$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^{-x^2} = 0$$

$\varepsilon \colon y = 0$

$\pm\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{-x^2} = 0$$

Πλαγιες

Αν εχει οριζοντια δεν εχει πλαγια.

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \alpha , \qquad \alpha \in \mathbb{R}^*$$

$$\lim_{x\to+\infty} f(x) - \alpha \cdot x = b$$

$$\varepsilon : y = \alpha x + b$$

$+\infty$.

7. (α) $f(x)=\dfrac{x^3}{x^2+1}$ $\quad D_f=\mathbb{R}$

Άρα δεν έχει κατακόρυφ. αφού $D_f=\mathbb{R}$.

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x^3}{x^2+1}=\lim_{x\to-\infty}\frac{x^3}{x^2}=$$

$$=\lim_{x\to-\infty} x=-\infty$$

$$\lim_{x\to+\infty} f(x)=+\infty$$

Δεν έχει οριζόντια.

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\frac{x^3}{x^2+1}}{x}=\lim_{x\to+\infty}\frac{x^3}{x^3+x}$$

$$=\lim_{x\to+\infty}\frac{x^3}{x^3}=1$$

$y=1\cdot x+0$

$y=x$

$+\infty$

$$\lim_{x\to+\infty} f(x)-1\cdot x=\lim_{x\to+\infty}\frac{x^3}{x^2+1}-x=$$

$$=\lim_{x\to+\infty}\frac{x^3-x(x^2+1)}{x^2+1}=\lim_{x\to+\infty}\frac{-x}{x^2+1}=\lim_{x\to+\infty}\frac{-\cancel{x}}{x^{\cancel{2}}}=0$$

7. ① $f(x) = \dfrac{e^x}{e^x+1}$ $\quad D_f = \mathbb{R}$

Δεν έχει κατακόρυφη.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{e^x}{e^x+1} = 0$$

$y = 0$
$-\infty$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^x}{e^x+1} =$$

$$= \lim_{x \to +\infty} \frac{e^x}{e^x} = 1$$

$y = 1$
$+\infty$

Προσοχή

$$f(x) = \begin{cases} xe^x, & x \le 0 \\ \ln x, & x > 0 \end{cases}$$

Η f δεν είναι συνεχής στο 0!

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln x = -\infty$$

$x=0$ κατακόρυφη

$$\lim_{x\to -\infty} xe^x = \lim_{x\to -\infty} \frac{x}{\frac{1}{e^x}} = \lim_{x\to -\infty} \frac{x}{e^{-x}}$$

$$= \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0.$$

$y=0$

$-\infty$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \ln x = +\infty$$

Δεν έχει οριζόντια

στο $+\infty$

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\ln x}{x}=\lim_{x\to+\infty}\frac{1}{x}=0$$

Δεν έχει πλάγια