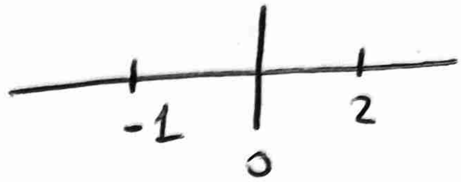
Ενοτητα 21

2. (γ) $f(x) = \begin{cases} x^2+x, & x<0 \\ x^3+x, & x\geq 0 \end{cases}$

Η f συνεχής $[-1,0)$ και $(0,2]$



ως π.σ.σ

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} x^2+x = 0 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x^3+x = 0 \end{aligned}\right\} \lim_{x\to 0} f(x) = 0$$

$f(0) = 0$

Συνεχής και στο 0!

✓ Συνεχής $[-1,2]$ ✓

Η f παρ/μη $(-1,0)$ και $(0,2)$ ως π.π.σ

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^2+x-0}{x} = \lim_{x\to 0^-} x+1 = 1$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^3+x}{x} = 1$$

✓ Η f παρ/μη στο 0. ✓

παρ/μη $(-1,2)$

Άρα $\exists\, \xi \in (-1,2)$ τ.ω $f'(\xi) = \frac{f(2)-f(-1)}{2-(-1)}$

$$f'(\xi) = \frac{10-0}{3}$$

$$f'(\xi) = \frac{10}{3}$$

$x < 0$

$f'(x) = \frac{10}{3}$

$2x+1 = \frac{10}{3}$

$6x+3 = 10$

$6x = 7$

$\cancel{x = \frac{7}{6}}$

$x \geq 0$

$f'(x) = \frac{10}{3}$

$3x^2+1 = \frac{10}{3}$

$9x^2+3 = 10$

$9x^2 = 7$

$x^2 = \frac{7}{9}$

$x = \frac{\sqrt{7}}{3}$

|

3. $f(4) - f(1) = 9$

(α) Νδο $\exists \xi \in (1,4)$ τ.ω. $f'(\xi) = 3$.

$$f'(\xi) = \frac{f(4)-f(1)}{4-1} = \frac{9}{3} = 3.$$

<u>Β' τρόπος</u>

$f'(x) = 3$

$f'(x) - 3 = 0$

$$\big(\underbrace{f(x) - 3x}_{g(x)}\big)' = 0$$

$g(1) = f(1) - 3 = f(4) - 9 - 3 = f(4) - 12$

$g(4) = f(4) - 12$

$g(1) = g(4)$

Rolle $\exists \xi \in (1,4)$ τ.ω. $g'(\xi) = 0$

$f'(\xi) - 3 = 0$

$f'(\xi) = 3$.

(B) Νδο $\exists M \in C_f$ τ.ω

η εφαπτομενη στο $M \parallel \varepsilon: y=3x+2$

$M(x_0, f(x_0))$

$y - f(x_0) = f'(x_0)(x - x_0) \parallel y = 3x-2$

Αρκει νδο $\exists x_0$ τ.ω $f'(x_0)=3$.

Το εδειξα πριν.

4. $f(1) < f(0)$

f δυο φορές παρ/μη

f συν + παρ.

f' συν + παρ

f''

α) Νδο $\exists\, \xi \in (0,1)$ τ.ω

$f'(\xi) < 0$

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0} < 0$$

β). Αφού $f'(x) \neq 0$ και συνεχής

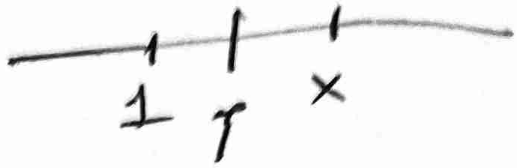
$\Rightarrow f'(x) > 0$ ή $f'(x) < 0$

$$f'(\xi) < 0$$

$$\underline{\underline{f'(x) < 0}}$$

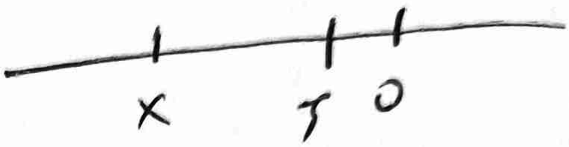
5. ⓐ $f'(\xi)=\dfrac{f(x)-f(1)}{x-1}$

$f(1)=0$



$$f'(\xi)=\frac{f(x)}{x-1}$$

$f(0)=0$

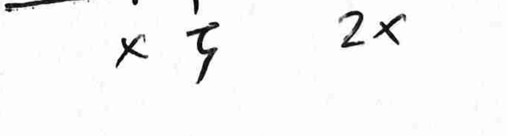


(β) $f'(\xi)=\dfrac{f(0)-f(x)}{0-x}$

$$f'(\xi)=\frac{-f(x)}{-x}$$

$$f'(\xi)=\frac{f(x)}{x}$$

$$x\,f'(\xi)=f(x)$$



(γ) $f'(\xi)=\dfrac{f(2x)-f(x)}{2x-x}$

$$f'(\xi)=\frac{f(2x)-f(x)}{x}$$

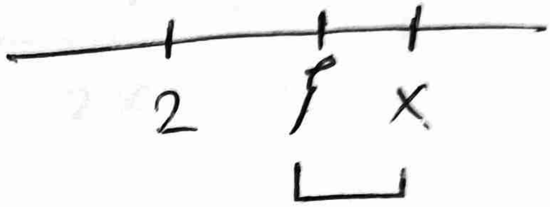
$$x\,f'(\xi)+f(x)=f(2x).$$

8. $f' \nearrow$

$f(2) = 0$

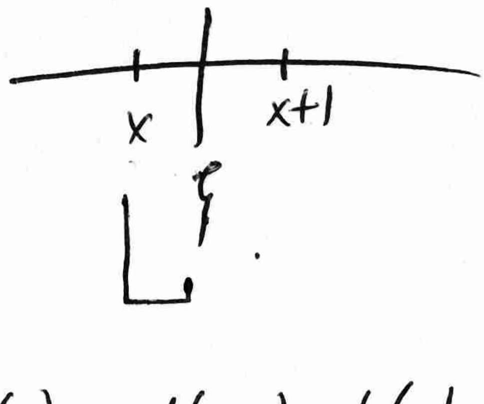
i) vdo $f'(x) > \dfrac{f(x)}{x-2}$ $\forall x > 2$

$$f'(\xi) = \frac{f(x) - f(2)}{x-2} = \frac{f(x)}{x-2}$$



$$\xi < x \overset{f' \nearrow}{\Rightarrow} f'(\xi) < f'(x) \Rightarrow \frac{f(x)}{x-2} < f'(x)$$

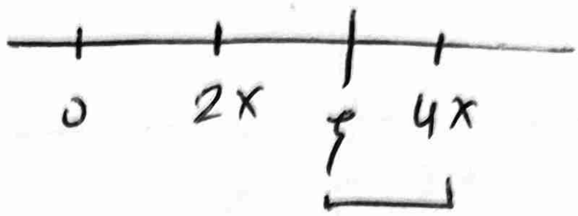
ii). vdo $f'(x) < f(x+1) - f(x)$



$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x}$$

$$x < \xi \Rightarrow f'(x) < f'(\xi) \Rightarrow \underline{\underline{f'(x) < f(x+1) - f(x)}}$$

iii) νδο $f(4x) - f(2x) < 2x\, f'(4x)$



$$f'(\xi) = \frac{f(4x)-f(2x)}{4x-2x} = \frac{f(4x)-f(2x)}{2x}$$

$$\xi < 4x \Rightarrow f'(\xi) < f'(4x) \Rightarrow \frac{f(4x)-f(2x)}{2x} < f'(4x)$$

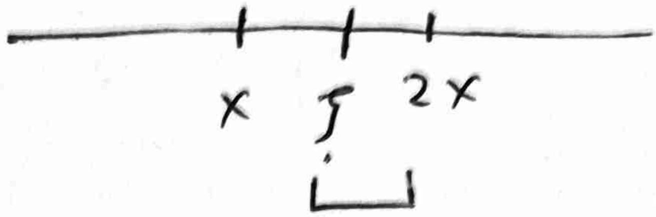
$$f(4x) - f(2x) < 2x\, f'(4x).$$

(B) הוכח $\sin 2x - \sin x > x\cos 2x$

$f(x) = \sin x$

$f'(x) = \cos x \quad f' \downarrow$

$f''(x) = -\sin x < 0$



$$f'(\xi) = \frac{f(2x) - f(x)}{2x - x} = \frac{\sin 2x - \sin x}{x}$$

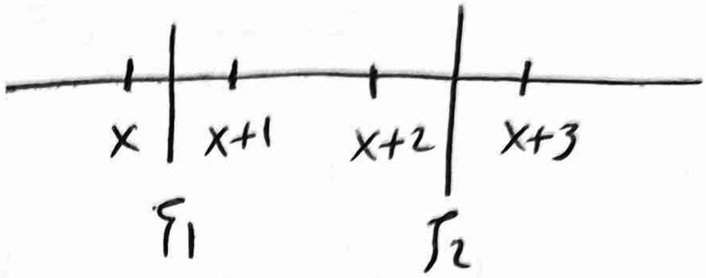
$$\xi < 2x \Rightarrow f'(\xi) > f'(2x)$$

$$\frac{\sin 2x - \sin x}{x} > \cos 2x$$

$$\boxed{\sin 2x - \sin x > x\cos 2x}$$

9. $f' \uparrow$

ⓐ Νδο $f(x+1) + f(x+2) < f(x) + f(x+3)$.



$$f'(\xi_1) = \frac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$$

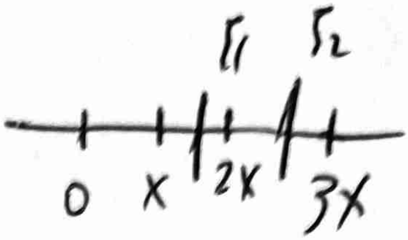
$$f'(\xi_2) = \frac{f(x+3) - f(x+2)}{x+3-x-2} = f(x+3) - f(x+2)$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$f(x+1) - f(x) < f(x+3) - f(x+2)$$

$$f(x+1) + f(x+2) < f(x) + f(x+3)$$

B) Νδο $f(x)+f(3x) > 2f(2x)$



$$f'(\xi_1)=\frac{f(2x)-f(x)}{2x-x}=\frac{f(2x)-f(x)}{x}$$

$$f'(\xi_2)=\frac{f(3x)-f(2x)}{3x-2x}=\frac{f(3x)-f(2x)}{x}$$

$$\xi_1<\xi_2 \Rightarrow f'(\xi_1)<f'(\xi_2)$$

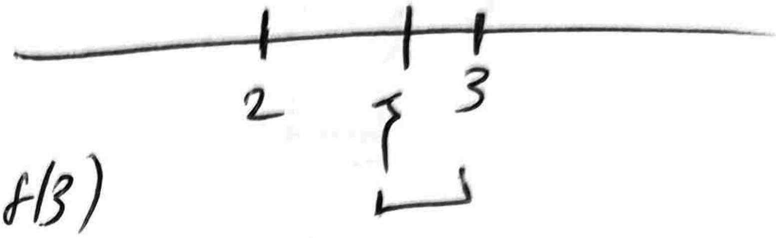
$$\frac{f(2x)-f(x)}{x}<\frac{f(3x)-f(2x)}{x}$$

$$\underline{\underline{2f(2x)<f(3x)+f(x)}}$$

10.

$f' \downarrow$

(a) Νδο $f'(3) < f(3)$ $\quad f(2) = 0$.

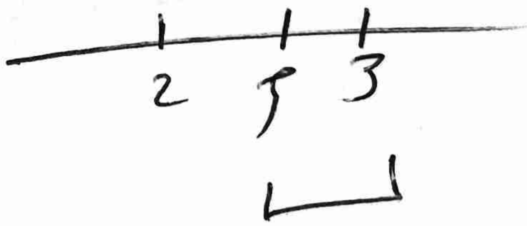


$$f'(\xi) = \frac{f(3) - f(2)}{3-2} = f(3)$$

$$\xi < 3 \Rightarrow f'(\xi) > f'(3)$$

$$f(3) > f'(3)$$

(a) Νδο $f'(3) + f(2) < f(3)$



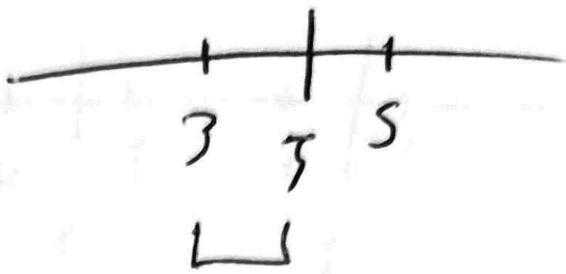
$$f'(\xi) = \frac{f(3) - f(2)}{3-2}$$

$$\xi < 3 \Rightarrow f'(\xi) > f'(3)$$

$$f(3) - f(2) > f'(3).$$

$$f(3) > f'(3) + f(2)$$ ✓

γ $f(5) < f(3) + 2f'(3)$.



$$f'(\xi) = \frac{f(5) - f(3)}{5-3}$$

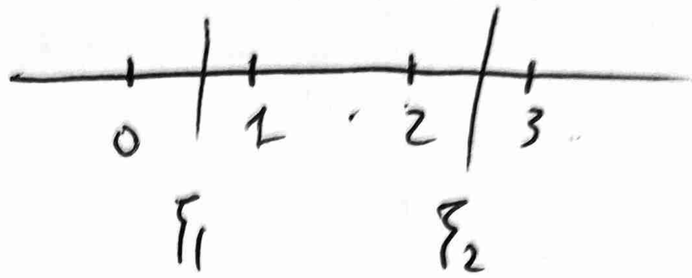
$$f'(\xi) = \frac{f(5) - f(3)}{2}$$

$$3 < \xi \Rightarrow f'(3) > f'(\xi)$$

$$f'(3) > \frac{f(5) - f(3)}{2}$$

$$2f'(3) + f(3) > f(5)$$

(δ) νδο $f(1)+f(2) > f(0)+f(3)$



$$f'(\xi_1) = \frac{f(1)-f(0)}{1-0} = f(1)-f(0)$$

$$f'(\xi_2) = \frac{f(3)-f(2)}{3-2} = f(3)-f(2)$$

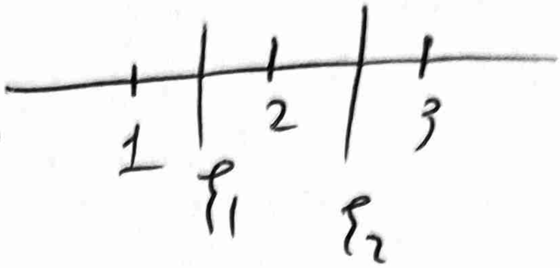
$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) > f'(\xi_2)$$

$$f(1)-f(0) > f(3)-f(2)$$

$$f(1)+f(2) > f(3)+f(0)$$

(ε) Νδο $f(1)+f(3) < 2f(2)$

$$f'(\xi_1) = \frac{f(2)-f(1)}{2-1} = f(2)-f(1)$$



$$f'(\xi_2) = \frac{f(3)-f(2)}{3-2} = f(3)-f(2)$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) > f'(\xi_2)$$

$$f(2)-f(1) > f(3)-f(2)$$

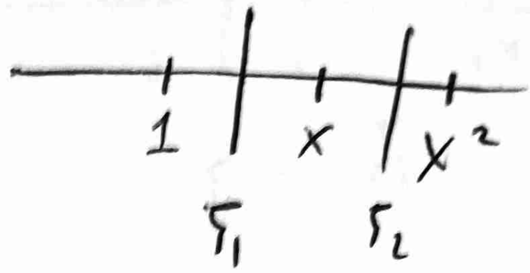
$$2f(2) > f(3)+f(1)$$

I

22.

$f' \uparrow$

ⓐ ΝΔΟ $(x+1)f(x) < x f(1) + f(x^2) \quad \forall x > 1$.



$$f'(\xi_1) = \frac{f(x) - f(1)}{x-1}$$

$$f'(\xi_2) = \frac{f(x^2) - f(x)}{x^2 - x}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$\frac{f(x) - f(1)}{\cancel{x-1}} < \frac{f(x^2) - f(x)}{x\cancel{(x-1)}}$$

$$x f(x) - x f(1) < f(x^2) - f(x)$$

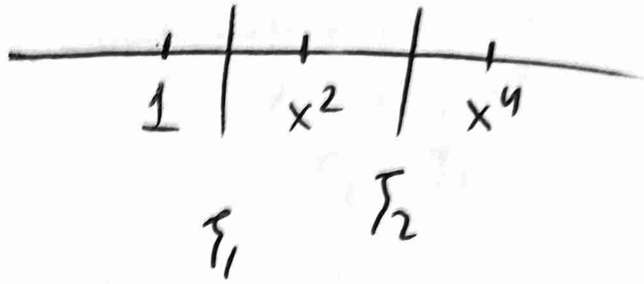
$$x f(x) + f(x) < f(x^2) + x f(1)$$

$$(x+1) f(x) < f(x^2) + x f(1)$$

(B) Νδο $(x^2+1)f(x^2) < x^2 f(1) + f(x^4)$

$x>1$

$$f'(\xi_1) = \frac{f(x^2)-f(1)}{x^2-1}$$



$$f'(\xi_1) = \frac{f(x^4)-f(x^2)}{x^4-x^2}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

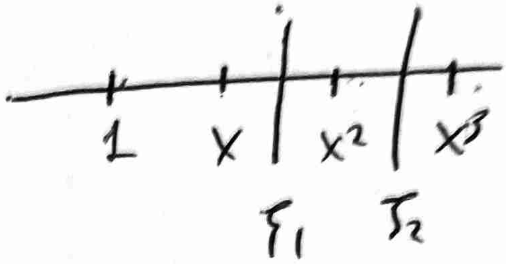
$$\frac{f(x^2)-f(1)}{\cancel{x^2-1}} < \frac{f(x^4)-f(x^2)}{x^2\cancel{(x^2-1)}}$$

$$x^2 f(x^2) - x^2 f(1) < f(x^4) - f(x^2)$$

$$x^2 f(x^2) + f(x^2) < f(x^4) + x^2 f(1)$$

$$(x^2+1)\ f(x^2) < f(x^4) + x^2 f(1).$$

(γ) νδο $x f(x) + f(x^3) > (x+1) f(x^2)$: $\forall x > 1$.



$$f'(\xi_1) = \frac{f(x^2) - f(x)}{x^2 - x}$$

$$f'(\xi_2) = \frac{f(x^3) - f(x^2)}{x^3 - x^2}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$\frac{f(x^2) - f(x)}{x^2 - x} < \frac{f(x^3) - f(x^2)}{x^3 - x^2}$$

$$\frac{f(x^2) - f(x)}{\cancel{x}\cancel{(x-1)}} < \frac{f(x^3) - f(x^2)}{x^{\cancel{2}}\cancel{(x-1)}}$$

$$x f(x^2) - x f(x) < f(x^3) - f(x^2) \Rightarrow (x+1) f(x^2) < x f(x) + f(x^3).$$

6. (8) Νδο $(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\dfrac{\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\beta} < (\beta-\alpha)\,\varepsilon\varphi\beta$

$0<\alpha<\beta<\frac{\pi}{2}$

$$\boxed{(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\sigma\upsilon\nu\alpha - \ln\sigma\upsilon\nu\beta < (\beta-\alpha)\,\varepsilon\varphi\beta}$$

$f(x)=\ln\sigma\upsilon\nu x$

$$f'(x)=\frac{-\eta\mu x}{\sigma\upsilon\nu x}$$

$$f''(x)=\frac{-\sigma\upsilon\nu^2x-(+\eta\mu^2x)}{\sigma\upsilon\nu^2x}=-\frac{1}{\sigma\upsilon\nu^2x}<0 \qquad f'\downarrow$$

$$f'(\xi)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=\frac{\ln\sigma\upsilon\nu\beta-\ln\sigma\upsilon\nu\alpha}{\beta-\alpha}$$

$$\alpha < \xi < \beta$$

$$f' \downarrow$$

$$f'(\alpha) > f'(\xi) > f'(\beta)$$

$$-\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha} > \frac{\ln\sigma\upsilon\nu\beta - \ln\sigma\upsilon\nu\alpha}{\beta-\alpha} > -\frac{\eta\mu\beta}{\sigma\upsilon\nu\beta}$$

$$-\varepsilon\varphi\alpha\,(\beta-\alpha) > \ln\sigma\upsilon\nu\beta - \ln\sigma\upsilon\nu\alpha > -\varepsilon\varphi\beta(\beta-\alpha)$$

$$\varepsilon\varphi\alpha\,(\beta-\alpha) < \ln\sigma\upsilon\nu\alpha - \ln\sigma\upsilon\nu\beta < \varepsilon\varphi\beta(\beta-\alpha)$$

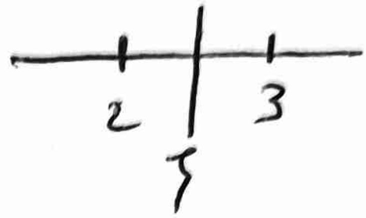
6. (δ) Νδο $\frac{1}{3} < \ln\frac{3}{2} < \frac{1}{2}$

$$\frac{1}{3} < \ln 3 - \ln 2 < \frac{1}{2}$$ ✓

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$ $\quad f' \cancel{\uparrow}$

$f''(x) = -\frac{1}{x^2} < 0$



$$f'(\xi) = \frac{f(3) - f(2)}{3-2} = \frac{\ln 3 - \ln 2}{1} = \ln 3 - \ln 2$$

$$2 < \xi < 3$$

$$f' \downarrow$$

$$f'(2) > f'(\xi) > f'(3)$$

$$\frac{1}{2} > \ln 3 - \ln 2 > \frac{1}{3}$$ ✓

7. Nδο $\dfrac{1}{x+1} < \ln\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}$ $\forall x > 0$.

α' τρόπος

- $\ln x \le x-1$

$$\ln\left(1+\frac{1}{x}\right) \le 1+\frac{1}{x}-1$$

$$\boxed{\ln\left(1+\frac{1}{x}\right) \le \frac{1}{x}}$$

- $\dfrac{1}{x+1} < \ln\left(1+\dfrac{1}{x}\right)$

$$\underbrace{\frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right)}_{g(x)} < 0$$

$$g'(x) = \frac{-1}{(x+1)^2} - \frac{-\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$g'(x) = -\frac{1}{(x+1)^2} + \frac{1}{x^2+1}$$

$$g'(x) = \frac{-x^2-1+(x+1)^2}{(x+1)^2(x^2+1)} = \frac{-\cancel{x^2}-\cancel{1}+\cancel{x^2}+2x+\cancel{1}}{(x+1)^2(x^2+1)}$$

$$g'(x) = \frac{2x}{(x+1)^2(x^2+1)} > 0 \quad \forall x>0$$

$$g \nearrow$$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right) = -\infty,$$

$$\lim_{x\to +\infty} g(x) = \lim_{x\to +\infty} \frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right) = 0$$

$$\Sigma T_g = (-\infty, 0) \Rightarrow g(x) < 0$$

$$\frac{1}{x+1} - \ln\left(1+\frac{1}{x}\right) < 0$$

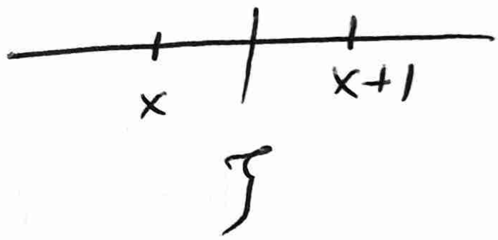
$$\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) \checkmark$$

В' тром 1

NБо $\frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$

$$\frac{1}{x+1} < \ln\left(\frac{x+1}{x}\right) < \frac{1}{x}$$

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$$



$f(x) = \ln x$

$f'(x) = \frac{1}{x}$ $\quad f' \downarrow$

$f''(x) = -\frac{1}{x^2} < 0$

$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x}$$

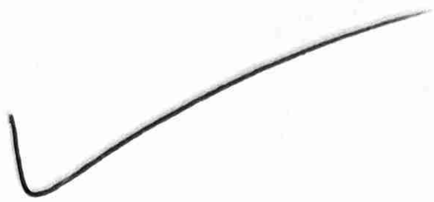
$$x < \xi < x+1$$

$$f' \downarrow$$

$$f'(x) > f'(\xi) > f'(x+1)$$

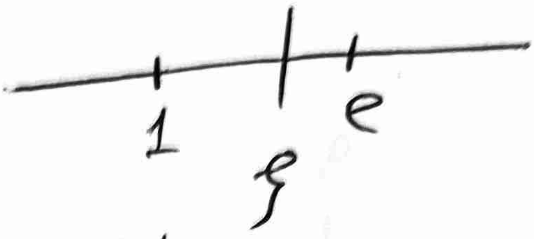
$$\frac{1}{x} > f(x+1) - f(x) > \frac{1}{x+1}$$

$$\frac{1}{x} > \ln(x+1) - \ln x > \frac{1}{x+1}$$



18. $f'(x) < \ln 3$ $\qquad f(e) = e\ln 3$

Νδο $f(1) > \ln 3$.



$$f'(\xi) = \frac{f(e)-f(1)}{e-1} = \frac{e\ln 3 - f(1)}{e-1}$$

ομως $f'(x) < \ln 3$

$$\Rightarrow f'(\xi) < \ln 3$$

$$\frac{e\ln 3 - f(1)}{e-1} < \ln 3$$

$$e\ln 3 - f(1) < (e-1)\ln 3$$

$$\cancel{e\ln 3} - f(1) < \cancel{e\ln 3} - \ln 3$$

$$f(1) > \ln 3 \quad \checkmark$$

19. $f''(x) \neq 0$

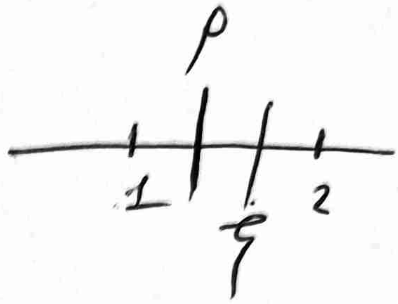
f שס+קצה.

f' שס+קצה

f'' שס

$\Rightarrow f''(x) > 0$

או $f''(x) < 0$



$$f'(\xi) = \frac{f(2) - f(1)}{2-1} = f(2) - f(1)$$

$$f(2) - f(1) > f'(1)$$

$$f'(\xi) > f'(1)$$

$$f''(p) = \frac{f'(\xi) - f'(1)}{\xi - 1} > 0$$

$\oplus$

$f''(p) > 0$ $\quad$ $f''(x) > 0$

Επομενο Μαθημα

21

6 α β

20

21

16

17

23

26.