31.

$f'(x) < 2x - \sin x$ $\qquad f(0) = 0$

(a) $f(x) > x^2 - \cos x \quad , \quad x < 0$

$$\underbrace{f(x) - x^2 + \cos x}_{g(x)} > 0$$

$$g'(x) = f'(x) - 2x + \sin x < 0$$

$g \downarrow$

$\forall x < 0 \Rightarrow g(x) > g(0) \Rightarrow f(x) - x^2 + \cos x > 0$

$$f(x) > x^2 - \cos x$$

(B) $f(x) = x^2 - \cos x$

$f(x) - x^2 + \cos x = 0$

$g(x) = 0$

$g(x) = g(0)$

g 1-1

$x = 0$

(γ) $f(x) > x^2 - \eta\mu x \quad \forall x<0$

$$\lim_{x\to-\infty} f(x) > \lim_{x\to-\infty} x^2 - \eta\mu x$$

$$\lim_{x\to-\infty} f(x) > +\infty$$

$$\lim_{x\to-\infty} f(x) = +\infty$$

$$-1 \le \eta\mu x \le 1$$

$$1 \geq -\eta\mu x \geq -1$$

$$x^2+1 \geq x^2-\eta\mu x \geq x^2-1$$

$$\left.\begin{array}{l} \lim\limits_{x\to-\infty} x^2+1 = +\infty \\ \lim\limits_{x\to-\infty} x^2-1 = +\infty \end{array}\right\} \lim_{x\to-\infty} x^2-\eta\mu x = +\infty$$

35. $f'(0) < 0$ $\qquad$ f συνεχής

$f'(x) \neq 0$ $\qquad$ f' συν.

ⓐ Αφού $f'(x) \neq 0$ και συνεχής

$\Rightarrow f'(x) > 0$ ή $f'(x) < 0$ $\forall x \in \mathbb{R}$.

$f'(0) < 0 \Rightarrow f'(x) < 0$

$f \downarrow$.

ⓑ $f(\varepsilon\varphi x) > f(\eta\mu x)$ $\quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$f \downarrow$

$\varepsilon\varphi x > \eta\mu x$

$\underbrace{\varepsilon\varphi x - \eta\mu x}_{g(x)} > 0 \Rightarrow g(x) > 0 \Rightarrow g(x) > g(0)$

$g \nearrow$

$$g'(x) = \frac{1}{\sigma\upsilon\nu^2 x} - \sigma\upsilon\nu x = \frac{1 - \sigma\upsilon\nu^3 x}{\sigma\upsilon\nu^2 x} > 0$$

(+) $\qquad$ $\frac{\pi}{2} > x > 0$

(+)

$g \nearrow$

$\sigma\upsilon\nu x \leq 1$

$\sigma\upsilon\nu^3 x \leq 1^3$

$\sigma\upsilon\nu^3 x \leq 1$

$0 \leq 1 - \sigma\upsilon\nu^3 x$

(γ) $f(x^2) - \ln x = f(x)$

$f(x^2) - \ln\frac{x^2}{x} = f(x)$

$f(x^2) - (\ln x^2 - \ln x) = f(x)$

$f(x^2) - \ln x^2 = f(x) - \ln x$

$\underline{\underline{h(x) = f(x) - \ln x}}$

$h(x^2) = h(x)$

h 1-1

$x^2 = x$

$x = 1$

$h'(x) = f'(x) - \frac{1}{x} < 0$

h

36. $f(x) = (x^2+1)e^x - 1$

(Β1) $f'(x) = 2xe^x + (x^2+1)e^x = e^x(2x+x^2+1)$

$$\boxed{f'(x) = e^x(x+1)^2 \geq 0}$$

$f \nearrow$

(Β) $f(e^x-1) + f(2x) = f(x)$

προφανώς $x=0$

$x>0$

- $x>0 \Rightarrow e^x > e^0 \Rightarrow e^x > 1 \Rightarrow e^x - 1 > 0$

  $f(e^x-1) > f(0)$

  $f(e^x-1) > 0$

- $x>0 \Rightarrow 2x > x \Rightarrow f(2x) > f(x)$

  $f(2x) - f(x) > 0$ (+)

  $f(e^x-1) + f(2x) - f(x) > 0$

$x<0$

- $x<0 \Rightarrow e^x < e^0 \Rightarrow e^x < 1 \Rightarrow e^x - 1 < 0 \Rightarrow f(e^x-1) < f(0)$

  $f(e^x-1) < 0$

- $x<0 \Rightarrow 2x < x \Rightarrow f(2x) < f(x)$ (+)

  $f(2x) - f(x) < 0$

  $f(e^x-1) + f(2x) - f(x) < 0$

38. $f' \nearrow$ $\qquad f(5) = f(4)$

ⓐ $\underbrace{f(x+1) - f(x)}_{g(x)} = 0$ $\quad \Rightarrow g(x) = 0$

$g(x) = g(4)$

g 1-1

$\underline{\underline{x = 4}}$

$g'(x) = f'(x+1) - f'(x)$

• $x < x+1 \rightarrow f'(x) < f'(x+1)$

$f'(x+1) - f'(x) > 0$

$g'(x) > 0$

$g \nearrow$

ⓑ $f(2x+1) > f(2x)$

$f(2x+1) - f(2x) > 0$

$g(2x) > g(4)$

$g \nearrow$

$2x > 4$

$x > 2.$

ⓒ $f(x^4+5) = f(x^4+4)$

$f(x^4+5) - f(x^4+4) = 0$

$g(x^4+4) = g(4)$

g 1-1

$x^4+4 = 4$

$x^4 = 0$

$x = 0.$

41. $f(x) = x^3 - 6\ln x$

$f'(x) = 3x^2 - \frac{6}{x}$

$y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(0,4)$

$4 - f(x) = f'(x)(0 - x)$

$4 - (x^3 - 6\ln x) = \left(3x^2 - \frac{6}{x}\right)(-x)$

$4 - x^3 + 6\ln x = -3x^3 + 6$

$2x^3 + 6\ln x - 2 = 0$

$\boxed{x^3 + 3\ln x - 1 = 0}$ $\quad \underline{x > 0}$

$\varphi(x) = 0$

$\varphi(x) = \varphi(1)$

$\varphi'(x) = 3x^2 + \frac{3}{x} > 0$

$\underline{x = 1}$

$\varphi\uparrow$

$\underline{y - f(1) = f'(1)(x - 1)}$

39. $f(x) = 2(x-2)\ln x + 2x^2 - 1 \quad , x > 0$

(01) $f'(x) = 2\ln x + 2\,\frac{x-2}{x} + 4x$

$f''(x) = \frac{2}{x} + 2\,\frac{x - x + 2}{x^2} + 4$

$f''(x) = \frac{2}{x} + \frac{4}{x^2} + 4 > 0$

$f' \nearrow$

(13) $f(x^3+2) + f(x^2) < f(x^2+2) + f(x^3)$

$f(x^3+2) - f(x^3) < f(x^2+2) - f(x^2)$

$$\boxed{g(x) = f(x+2) - f(x)}$$

$g'(x) = f'(x+2) - f'(x)$

$g(x^3) < g(x^2)$

g 1-1

- $x < x+2$

$f' \nearrow$

$f'(x) < f'(x+2)$

$\underline{f'(x+2) - f'(x) > 0}$

$x^3 < x^2$

$\underline{\underline{0 < x < 1}}$

$g'(x) > 0 \qquad g \nearrow$

43. $f(x) = \ln(x-2) + 4$ $\qquad g(x) = e^{x} - x^2$

$\underline{\underline{x > 2}}$

(a) $y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow A(0,1)$

$1 - f(x) = f'(x)(0 - x).$

$$1 - (\ln(x-2) + 4) = \frac{1}{x-2}(-x)$$

$$1 - \ln(x-2) - 4 = -\frac{x}{x-2}$$

$$-3 - \ln(x-2) = -\frac{x}{x-2}$$

$$3 + \ln(x-2) = \frac{x}{x-2}$$

$$\underbrace{3 + \ln(x-2) - \frac{x}{x-2}}_{\varphi(x)} = 0$$

$\varphi(x) = \varphi(3)$

φ 1-1

$\underline{\underline{x = 3}}$

$$\varphi'(x) = \frac{1}{x-2} - \frac{x-2-x}{(x-2)^2}$$

$$\varphi'(x) = \frac{1}{x-2} + \frac{2}{(x-2)^2} > 0 \quad \varphi \nearrow.$$

(B) $y - f(3) = f'(3)(x-3)$

$y - 4 = x - 3$

$y = x + 1$

$$\begin{cases} g(x) = x+1 \\ g'(x) = 1 \end{cases}$$

Αρκει νδο εχει μοναδική λυση

$e^x - x^2 = x+1 \longrightarrow 2x+\cancel{1} - x^2 = x+\cancel{1}$

$e^x - 2x = 1$

$-x^2 + x = 0$

$e^x = 2x+1$

$x(-x+1) = 0$

$x = 0$ ✓

$\cancel{x=1}$

Ενότητα 24

f συν + παρ
f' συν + παρ
f''

24. $f(1) = f(2) = 0$ $\qquad f''(x) < 0$

(α) Αφού f συνεχής $[1,2]$

f παρ/μη $(1,2)$

$f(1) = f(2) = 0$

Rolle $\exists\, \xi \in (1,2)$ τ.ω $f'(\xi) = 0$.

Αφού $f''(x) < 0$

$f' \downarrow$ το ξ μοναδικό.

(β). $f(x) = 0$

Προφανείς ρίζες οι $x=1$, $x=2$

x		ξ	
f''	$-$		$-$
f'	$\searrow$ $+$	0	$-$ $\searrow$
f	$\nearrow$		$\searrow$

$x < \xi \Rightarrow f'(x) > f'(\xi)$
$\Rightarrow f'(x) > 0$

$x > \xi \Rightarrow f'(x) < f'(\xi)$
$f'(x) < 0$

2 διαστήματα μονοτονίας άρα 2 ρίζες.

Ενοτητα 20

2. (B) $f(x) = \dfrac{e^x}{x}$ $\quad \Delta = [\ln 2, \ln 4]$

✓ Η $f(x)$ συνεχής στο $[\ln 2, \ln 4]$ ως π.σ.σ

✓ Η $f(x)$ παρ/μη στο $(\ln 2, \ln 4)$ ως π.π.σ.

$$f(\ln 2) = \frac{e^{\ln 2}}{\ln 2} = \frac{2}{\ln 2}$$

$$f(\ln 4) = \frac{e^{\ln 4}}{\ln 4} = \frac{4}{\ln 4} = \frac{4}{\ln 2^2} = \frac{4}{2\ln 2} = \frac{2}{\ln 2}$$

✓ Άρα $f(\ln 2) = f(\ln 4)$

$\exists\ \xi \in (\ln 2, \ln 4)$ τ.ω $f'(\xi) = 0$.

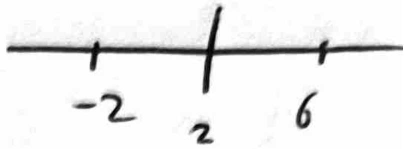
$$f'(x) = \frac{e^x x - e^x}{x^2}$$

$$f'(\xi) = 0 \Leftrightarrow \frac{e^\xi \xi - e^\xi}{\xi^2} = 0 \Rightarrow e^\xi \xi - e^\xi = 0$$

$$\xi - 1 = 0$$

$$\underline{\xi = 1}$$

16. (α) $f(x) = \begin{cases} x^2-6x+3, & x>2 \\ -2x-1, & x\le 2 \end{cases}$ $\quad Δ = [-2, 6)$



$$\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= -5 \\ \lim_{x\to 2^+} f(x) &= -5 \end{aligned}\right\} \lim_{x\to 2} f(x) = -5$$

$f(2) = -5$ $\quad$ Συνεχής στο 2

Επίσης η f συνεχής στο $[-2,2)$ και $(2,6]$ ως π.σ.σ

✓ Άρα συνεχής $[-2,6]$.

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{-2x-1+5}{x-2} = \lim_{x\to 2^-} \frac{-2x+4}{x-2}$$

$$= \lim_{x\to 2^-} \frac{-2}{1} = -2$$

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{x^2-6x+3+5}{x-2}$$

$$= \lim_{x\to 2^+} \frac{x^2-6x+8}{x-2} = \lim_{x\to 2^+} \frac{2x-6}{1} = -2$$

Άρα παρ/μη στο 2.

Επομένως η f παρ/μη στο $(-2,2)$

και $(2,6]$ ω π.ο. σ.

Άρα η f παρ/μη $(-2,6)$

$f(-2)=3$

$f(6)=3$

ισχύει ο Rolle.

$\exists\, \xi \in (-2,6)$ τ.ω $f'(\xi)=0$

$x \leq 2$

$f'(x)=-2 \neq 0$

$x>2$

$f'(x)=2x-6$

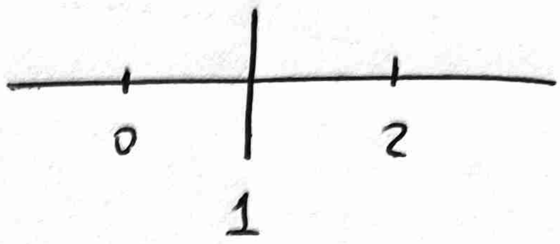
$f'(\xi)=0$

$2\xi-6=0$

$\xi=3$

3. (β) $f(x) = \begin{cases} e^x, & x<1 \\ x-1, & x \geq 1 \end{cases}$ $\quad \Delta = [0,2]$

$\lim\limits_{x\to 1^-} f(x) = e$
$\lim\limits_{x\to 1^+} f(x) = 0$ $\Big\}$ όχι συνεχής στο 1



Άρα δεν ικανοποιείται ο Rolle.

7. $f: [0,\pi] \to \mathbb{R}$ συνεχής στο $[0,\pi]$
παρ/μη $(0,\pi)$

(α) Ν.δ.ο $g(x) = f(x)\,ημx$ ικανοποιεί Rolle $[0,\pi]$

✓ Η $g(x)$ συνεχής $[0,\pi]$ ως π.σ.σ

✓ Η $g(x)$ παρ/μη $(0,\pi)$ ως π.π.σ

$g(0) = 0$

✓ $g(\pi) = 0$

Rolle $\exists\, \xi \in (0,\pi)$ που $g'(\xi) = 0$

(B) Νδο η εξισωση $f'(x) = -f(x)\,\sigma\varphi x$

έχει ρίζα $(0,\pi)$

$$g'(x) = f'(x)\,\eta\mu x + f(x)\,\sigma\upsilon\nu x$$

$$g'(\xi) = f'(\xi)\,\eta\mu\xi + f(\xi)\,\sigma\upsilon\nu\xi = 0$$

$$f'(\xi) + f(\xi)\,\frac{\sigma\upsilon\nu\xi}{\eta\mu\xi} = 0$$

$$f'(\xi) + f(\xi)\,\sigma\varphi\xi = 0$$

$$f'(\xi) = -f(\xi)\,\sigma\varphi\xi$$

6. $f(x) = \sigma\upsilon\nu x \cdot \ln x$

(α) Η $y - f(\xi) = f'(\xi)(x-\xi)$ // $x'x$

Αρκει νδο $\exists \xi \in (1, \frac{\pi}{2})$ τ.ω

$$\underline{f'(\xi) = 0}$$

Η f συν. ν $\pi.\delta.\sigma$

Η f παρ. ν $\pi.\pi.\delta$

$f(1) = 0$

$f(\frac{\pi}{2}) = 0$

$\Big\}$ Rolle $\exists \xi \in (1, \frac{\pi}{2})$ τ.ω $f'(\xi) = 0$.

(β) Νδο η εξισωση $\sigma\varphi x = x \ln x$ εχει 1 ριζα $(1, \frac{\pi}{2})$.

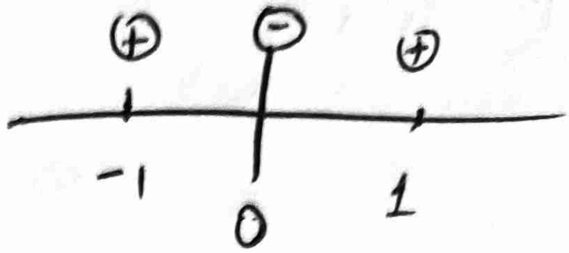
$f'(x) = -\eta\mu x \ln x + \sigma\upsilon\nu x \frac{1}{x}$

$f'(\xi) = 0 \iff -\eta\mu\xi \ln\xi + \frac{\sigma\upsilon\nu\xi}{\xi} = 0$

$-\ln\xi + \frac{\sigma\upsilon\nu\xi}{\eta\mu\xi} \frac{1}{\xi} = 0$

$\sigma\varphi\xi \frac{1}{\xi} = \ln\xi$ $\qquad \underline{\sigma\varphi\xi = \xi\ln\xi}$

11. ⓐ $2x^4 + \lambda^2(x^2-1) = x$, $\lambda \neq 0$ $(-1,1)$



$$\boxed{\varphi(x) = 2x^4 + \lambda^2(x^2-1) - x}$$

$\varphi(-1) = 2 + 1 = 3 > 0$

$\varphi(0) = -\lambda^2 < 0$

$\varphi(1) = 1 > 0$

$\varphi(-1)\varphi(0) < 0$ Bolzano $\exists \xi_1 \in (-1,0)$ τ.ω $\varphi(\xi_1) = 0$

$\varphi(0)\varphi(1) < 0$ Bolzano $\exists \xi_2 \in (0,1)$ τ.ω $\varphi(\xi_2) = 0$

β) $8x^3 + 2\lambda^2 x = 1$ $(-1,1)$.

Αφού $\varphi(\xi_1) = \varphi(\xi_2)$

Rolle $\exists \xi \in (\xi_1, \xi_2)$ τ.ω $\varphi'(\xi) = 0$

$\varphi'(x) = 8x^3 + 2x\lambda^2 - 1$

$\varphi'(\xi) = 8\xi^3 + 2\xi\lambda^2 - 1 = 0 \Rightarrow \underline{8\xi^3 + 2\xi\lambda^2 = 1}$

13. $f: \mathbb{R} \to \mathbb{R}$ παρ/μη.

$f'(x) \neq -1 \quad \forall x \in \mathbb{R}$

· <u>κοινά σημεία</u>

$f(x) = -x$

$f(x) + x = 0 \Rightarrow g(x) = 0$

Έστω ότι η $g(x)$ έχει

2 ρίζα.

$g(x_1) = g(x_2) = 0$

Rolle $\exists\, \xi \in (x_1, x_2)$

τ.ω $g'(\xi) = 0$

$g'(x) = f'(x) + 1$

$g'(\xi) = f'(\xi) + 1 = 0 \Rightarrow f'(\xi) = -1$

άρα έχει το πολύ 1 ρίζα. Άτοπο!

15. $f(x) = 3^x + x^2 - 3x - 1$

α) Η f συνεχής $[0,1]$

Η f παρ/μη $[0,1]$

$f(0) = 0$

$f(1) = 0$

Rolle $\exists\, \xi \in (0,1)$ τ.ω. $f'(\xi) = 0$.

β) Έστω ότι η f έχει 3 ρίζες

$$\underbrace{f(x_1) = f(x_2)}_{\text{Rolle}} \quad \underbrace{= f(x_3)}_{\text{Rolle}} = 0$$

$$\underbrace{f'(\xi_1) = f'(\xi_2)}_{\text{Rolle}} = 0$$

$$f''(\xi) = 0$$

$f'(x) = 3^x \ln 3 + 2x - 3$

$f''(x) = 3^x \ln^2 3 + 2$

$f''(\xi) = 3^{\xi} \ln^2 3 + 2 = 0$

Άτοπο!

Άρα έχει ακριβώς 2 ρίζες

(8) $g(x) = 3^x$ $h(x) = -x^2+3x+1$

$g(x) = h(x)$

$3^x = -x^2+3x+1$

$3^x + x^2 - 3x - 1 = 0$

$f(x) = 0$

$x = 0$ $x = 1$

23. $f: \mathbb{R} \to \mathbb{R}$ παρ/μη

$$f(2) = f(3) = 0$$

α) $G(x) = \dfrac{f(x)}{x-4}$ $\quad [2,3]$

Η $G(x)$ συνεχής $[2,3]$ ως π.σ.σ

Η $G(x)$ παρ/μη $(2,3)$ ως π.π.σ.

$$G(2) = \frac{f(2)}{-2} = 0$$

$$G(3) = \frac{f(3)}{-1} = 0$$

Rolle $\exists\, \xi \in (2,3)$ τ.ω $G'(\xi) = 0$,

β) $y - f(\xi) = f'(\xi)(x - \xi) \longrightarrow A(4,0)$

$$0 - f(\xi) = f'(\xi)(4 - \xi)$$

$$\boxed{-f(\xi) = f'(\xi)(4-\xi)}$$

$$G(x) = \frac{f(x)}{x-4}$$

$$G'(x) = \frac{f'(x)(x-4) - f(x)}{(x-4)^2}$$

$$G'(\xi) = \frac{f'(\xi)(\xi-4) - f(\xi)}{(\xi-4)^2} = 0$$

$$f'(\xi)(\xi-4) - f(\xi) = 0$$

$$-f(\xi) = -f'(\xi)(\xi-4)$$

$$-f(\xi) = f'(\xi)(4-\xi)$$

Επομενο Μαθημα

24	20
45	2 α
46	3 α
47	4
48	5
49	8
52	9
56	10
57	12
58	14
59	24
60	