Θεμα

Εστω $f(x) = \begin{cases} e^{1-x} - \ln x + 1 , & x \geq 1 \\ x^3 + x , & x < 1 . \end{cases}$

1. Ειναι συνεχης;

2. Ειναι παραγωγισιμη;

3. Οριζεται εφαπτομενη στο 1;
Αν ναι βρες την.

4. Εφαπτομενη στο 0!

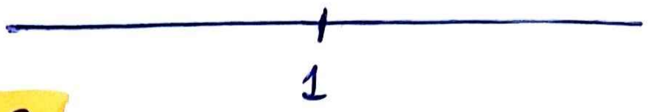
5. Συνολο Τιμων της $f(x)$.

6. Πληθος ριζων της εξισωσης $f(x)=\lambda$.

7. Νδο η εξισωση $e^{f(x)} + f(x) = 1$
εχει ακριβως δυο ριζες στο R.

Λυση

1. $f(x) = \begin{cases} e^{1-x} - \ln x + 1, & x \geq 1 \\ x^3 + x, & x < 1. \end{cases}$



- $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^3+x) = 2$

- $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} e^{1-x} - \ln x + 1 = e^0 - 0 + 1 = 2$

Άρα $\lim_{x\to1} f(x) = 2$ ενώ $f(1) = 2$.

Συνεχής στο 1!.

Η f(x) είναι συνεχής στο $(-\infty, 1)\cup(1, +\infty)$ ως π. δ. σ.

2. $\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{x^3+x-2}{x-1} \underset{DLH}{\overset{(\frac{0}{0})}{=}} \lim_{x\to1^-} \frac{3x^2+1}{1} = 4.$

$\lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{e^{1-x} - \ln x + 1 - 2}{x-1} = \lim_{x\to1^+} \frac{e^{1-x} - \ln x - 1}{x-1} \underset{DLH}{\overset{(\frac{0}{0})}{=}}$

$$= \lim_{x\to 1^+} \frac{-e^{1-x} - \frac{1}{x}}{1} = -1-1 = -2.$$

Αρα η $f(x)$ δεν ειναι παρ/μη στο 1.

3. Αφου το $f'(1)$ δεν υπαρχει

δεν οριζεται η εφαπτομενη στο 1.

4. $y - f(0) = f'(0)(x-0)$

$$\boxed{\begin{array}{l} f(x) = x^3+x \\ f'(x) = 3x^2+1 \end{array}}$$

- $f(0) = 0$
- $f'(0) = 1$

$y - 0 = 1\cdot(x-0)$

$$\boxed{y = x}$$

5. $D_f = \mathbb{R}$.

$x < 1$

$f_1(x) = x^3 + x$

- $x_1 < x_2 \Rightarrow x_1^3 < x_2^3$
- $x_1 < x_2$ $\oplus$

$$\underbrace{x_1^3 + x_1}_{f(x_1)} < \underbrace{x_2^3 + x_2}_{f(x_2)}$$

$f \nearrow$

$x \geq 1$

$f_2(x) = e^{1-x} - \ln x + 1$

- $x_1 < x_2 \Rightarrow -x_1 > -x_2$

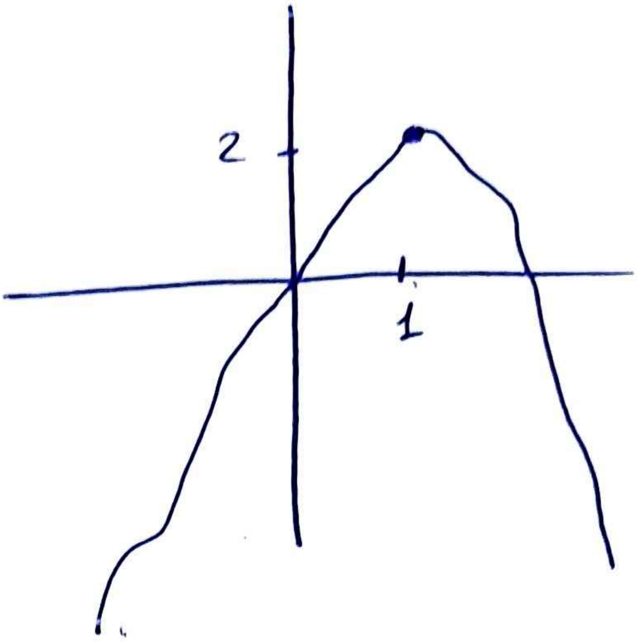
$1 - x_1 > 1 - x_2$

$e^{1-x_1} > e^{1-x_2}$ $\oplus$

- $x_1 < x_2 \Rightarrow -\ln x_1 + 1 > -\ln x_2 + 1$

$f \searrow$

x	$-\infty$	1	$+\infty$
$f(x)$	$-\infty \nearrow 2$	$\|\|$	$\searrow -\infty$



- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}(x^3+x) = -\infty$

- $f(1) = 2$

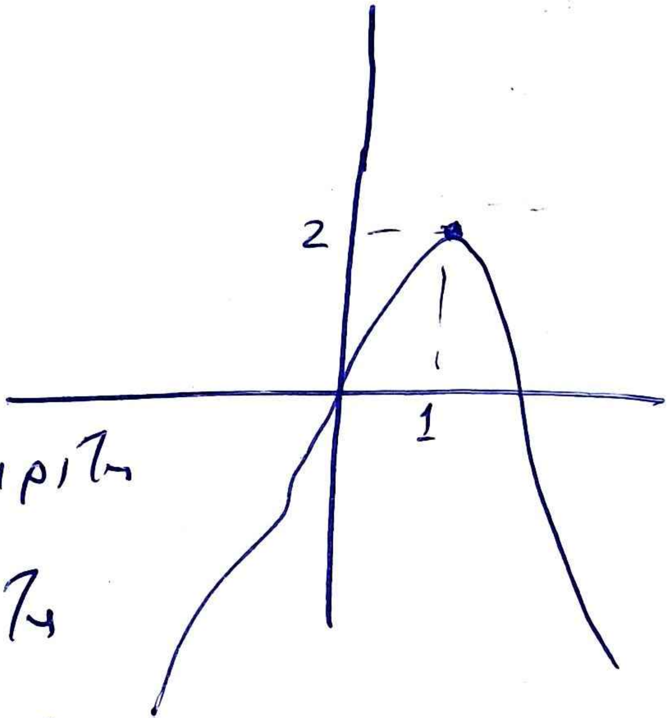
- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left(e^{1-x} - \ln x + 1\right) = e^{-\infty} - \ln(+\infty) + 1$

$= 0 - (+\infty) + 1 =$

$= -\infty$

$\Sigma T_f = (-\infty, 2]$

$f(x) \leq 2$



6. $f(x) = \alpha$.

Αν $\alpha > 2$ καμια ριζα

Αν $\alpha = 2$ 1 ριζα

Αν $\alpha < 2$ 2 ριζες.

7. $e^{f(x)} + f(x) = 1$ $\longrightarrow$ $\varphi(f(x)) = \varphi(0)$

φ 1-1

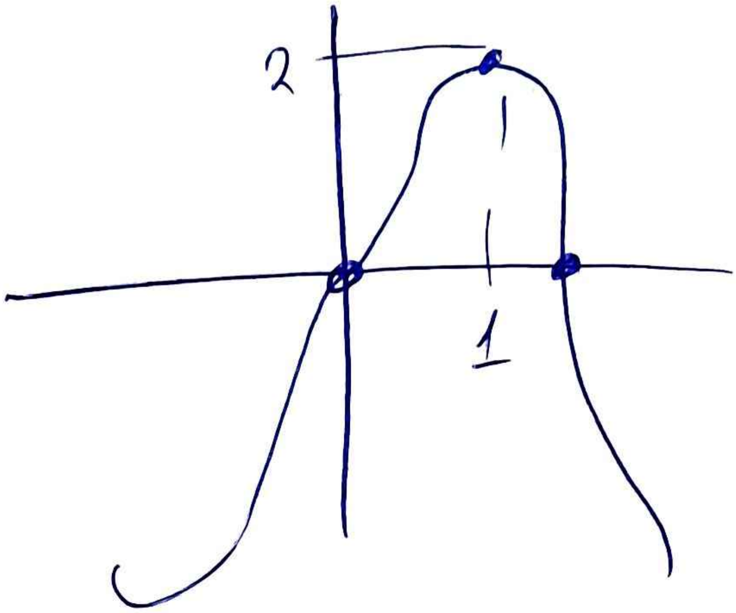
$f(x)=0$.

$\boxed{\varphi(x)=e^x+x}$

- $x_1<x_2 \Rightarrow e^{x_1}<e^{x_2}$
- $x_1<x_2$ ⊣ ⊕

$\varphi \nearrow$

φ 1-1



Αρκει νδο η εξισωση $f(x)=0$ εχει ακριβως δυο ριζες δηλ. αρκει νδο η C_f τεμνει τον $x'x$ 2 φορες.

$x<1$

- f συνεχης
- $f \nearrow$
- $\Sigma T_f = (-\infty, 2]$

Το $0 \in \Sigma T_f$ αρα $\exists! \xi_1<1$ τ.ω $f(\xi_1)=0$

$x \geq 1$

- f συνεχης
- $f \searrow$
- $\Sigma T_f = (-\infty, 2]$

Το $0 \in \Sigma T_f$ αρα $\exists! \xi_2>1$ τ.ω $f(\xi_2)=0$

Σελ. 338

2. ⓐ $f(x)=x^2-3x+1$ $\quad x_0=2$

$y-f(2)=f'(2)(x-2) \longrightarrow y+1=x-2$

- $f(2)=2^2-6+1=4-6+1=-1.$

$\boxed{y=x-3}$

- $f'(x)=2x-3$
- $f'(2)=2\cdot 2-3=1.$

Ⓑ. $y-f(0)=f'(0)(x-0)$ $\quad f(x)=e^x$

$y-1=x-0$ $\quad \boxed{y=x+1}$ $\quad f'(x)=e^x$

- $f(0)=e^0=1$
- $f'(0)=e^0=1.$

7. $y-f(x_0)=f'(x_0)(x-x_0) \longrightarrow M(-1,0).$

$f(x)=x^2-x-1$

$f'(x)=2x-1.$

$0-f(x_0)=f'(x_0)(-1-x_0)$

$0-f(x)=f'(x)(-1-x).$

$-(x^2-x-1)=(2x-1)(-1-x)$

$y-f(0)=f'(0)(x-0)$

$y-f(-2)=f'(-2)(x+2)$

$-x^2+x+1=-2x-2x^2+1+x$

$x^2+2x=0$ $\quad \boxed{x=0}$ $\quad \boxed{x=-2}$

10. $\boxed{f(x)=\frac{x^2}{2}-x+1 .}$

ⓐ $y-f(x_0)=f'(x_0)(x-x_0)$

$\boxed{f'(x)=x-1}$

$f'(x_0)=2$

$f'(x)=2$

$x-1=2$

$\boxed{x=3}$

$\boxed{y-f(3)=f'(3)(x-3)}$

Ⓑ i) $\varepsilon\varphi\omega<2$

$f'(x)<2$

$x-1<2$

$x<3$

ii). $\varepsilon\varphi 150=f'(x)$

$f'(x)=-\frac{\sqrt{3}}{3}$

$x-1=-\frac{\sqrt{3}}{3}$

$x=1-\frac{\sqrt{3}}{3}$

iii) $\varepsilon\varphi\hat{\omega}<0$

$f'(x)<0$

$x-1<0$

$x<1$

11. $f(x) = x^2 - x + 2.$ $f'(x) = 2x-1$

ⓐ $y - f(x_0) = f'(x_0)(x - x_0) \parallel y = 3x - 1.$

$f'(x) = 3$

$2x - 1 = 3$

$2x = 4$

$\boxed{x = 2}$ $A(2, 4)$

Ⓑ $y - f(x_0) = f'(x_0)(x - x_0) \perp \varepsilon_{AB}$

$f'(x_0) \cdot \lambda_{AB} = -1$

$(2x-1) \cdot \frac{f(2) - f(0)}{2 - 0} = -1$

$(2x-1) \cdot \frac{4 - 2}{2} = -1$

$(2x-1) = -1$

$2x = 0$

$\boxed{x = 0}$ $B(0, 2).$

(γ) $y - f(x_0) = f'(x_0)(x - x_0)$ $//$ $x'x$

$f'(x_0) = 0$.

$2x - 1 = 0$

$x = \frac{1}{2}$

$\Gamma\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right).$

12. $f(x) = \alpha x^3 + \beta x - 1$

$\varepsilon : y - f(1) = f'(1)(x-1) \perp \delta : 3x - y + 1 = 0$

$\boxed{y = 3x+1}$

$f'(1) \cdot \lambda_\delta = -1$

$(3\alpha + \beta) \cdot 3 = -1$

$\boxed{9\alpha + 3\beta = -1}$

$f(1) = -1$

$\alpha + \beta - 1 = -1$

$\boxed{\alpha + \beta = 0}$

$\boxed{f'(x) = 3\alpha x^2 + \beta}$

$$\begin{cases} 9\alpha + 3\beta = -1 \\ \alpha = -\beta \end{cases} \Rightarrow -9\beta + 3\beta = -1$$

$-6\beta = -1$

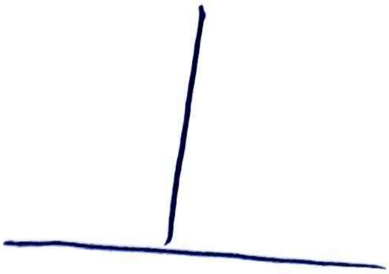
$\boxed{\beta = \frac{1}{6}}$

$\boxed{\alpha = -\frac{1}{6}}$

14. $f(x) = x^3 + ax^2 - x - 1$

ⓐ $y - f(0) = f'(0)(x-0)$

$y - f(1) = f'(1)(x-1)$



$$\boxed{f'(0) \cdot f'(1) = -1}$$

$f'(x) = 3x^2 + 2ax - 1$

$$-1 \cdot (3 + 2a - 1) = -1$$

$$3 + 2a - 1 = 1$$

$$2a = -1$$

$$\boxed{a = -\frac{1}{2}}$$

$$\boxed{f(x) = x^3 - \frac{1}{2}x^2 - x - 1}$$

$$\boxed{f'(x) = 3x^2 - x - 1}$$

Ⓑ. $εφ\hat{ω} = f'(0) = -1$

$ω = 135$

$εφ\hat{φ} = f'(1) = 1$

$\hat{φ} = 45$

15. $f(x) = \alpha \ln x + \beta x^2 - \ln \alpha$

$y - f(1) = f'(1)(x-1)$ εχει κλιση 3 .

$\underline{\underline{f'(1) = 3}}$.

$\underline{\underline{f(1) = 1}}$

$$f'(x) = \frac{\alpha}{x} + 2\beta x$$

$$f'(1) = \boxed{\alpha + 2\beta = 3}$$

$$f(1) = \boxed{\beta - \ln \alpha = 1}$$

$$\beta = 1 + \ln \alpha$$

$$\alpha + 2(1 + \ln \alpha) = 3$$

$$\alpha + 2 + 2\ln \alpha = 3$$

$$\alpha + 2\ln \alpha - 1 = 0 .$$

$$\boxed{\varphi(x) = x + 2\ln x - 1}$$

$\varphi \nearrow$

$$\varphi(\alpha) = \varphi(1)$$

φ 1-1

$\underline{\underline{\alpha = 1}}$

$\underline{\underline{\beta = 1}}$

20. $f(x) = x^2 - x + \frac{5}{4}$

$g(x) = e^{-2x}$

$$\boxed{y - f(-\tfrac{1}{2}) = f'(-\tfrac{1}{2})(x + \tfrac{1}{2})}$$

$f(-\frac{1}{2}) = \frac{1}{4} + \frac{1}{2} + \frac{5}{4} = \frac{6}{4} + \frac{2}{4} = 2$

$f'(x) = 2x - 1$

$f'(-\frac{1}{2}) = -2$

$y - 2 = -2(x + \frac{1}{2}) \Rightarrow y = -2x - 1 + 2$

$$\boxed{y = -2x + 1}$$

Αρκει νδο η $y = -2x + 1$ εφαπτεται της C_g

$$\begin{cases} g'(x) = -2 \\ g(x) = -2x + 1 \end{cases} \Leftrightarrow \begin{cases} -2e^{-2x} = -2 \Rightarrow e^{-2x} = 1 \quad \boxed{x = 0} \\ e^{-2x} = -2x + 1 \end{cases}$$

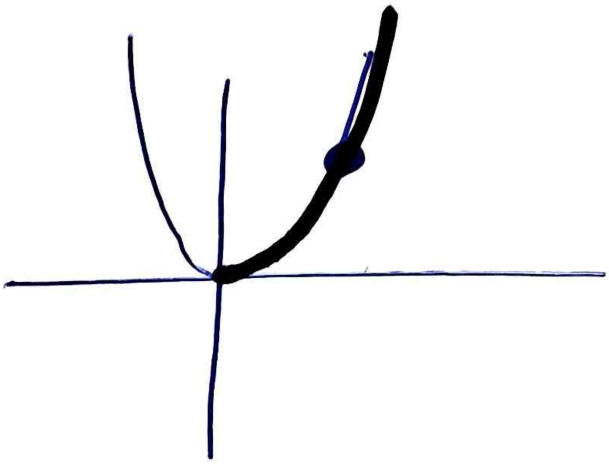
$e^0 = 0 + 1$

$e^0 = 1$ ✓

Οντως εφαπτεται στο 0!

Εστω συναρτηση $f(x) = 2x^2,\ x \geq 0$

Εστω $x'(t) = 2$

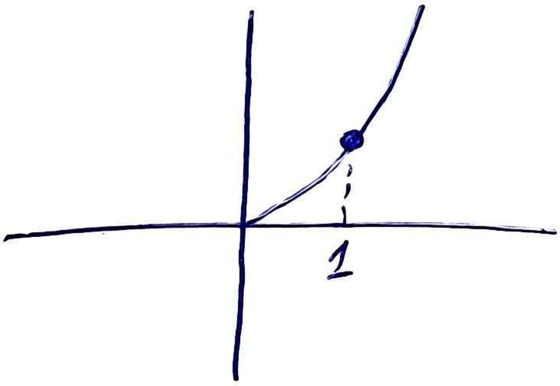


ⓐ Βρει το ρυθμο μεταβολης της τεταγμενης του υλικου σημειου τη χρονικη στιγμη που η τετμημενη ειναι 1.

$y = 2x^2$

$$\boxed{y(t) = 2x^2(t)}$$

$y'(t) = 2 \cdot 2x(t)\, x'(t)$



Εστω t_1 η χρονικη στιγμη που η τετμημενη ειναι 1 $\Rightarrow$ $x(t_1) = 1$, $y(t_1) = 2$.

$y'(t_1) = 2 \cdot 2x(t_1)\, x'(t_1)$

$y'(t_1) = 4\, x(t_1)\, x'(t_1)$

$y'(t_1) = 4 \cdot 1 \cdot 2$

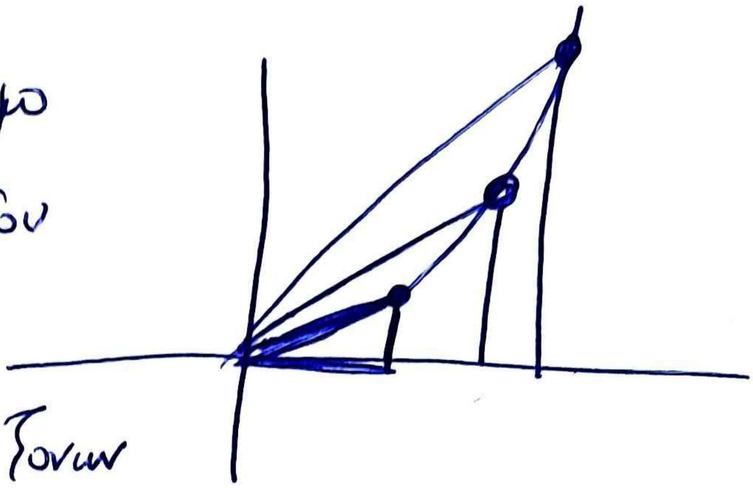
$y'(t_1) = 8$.

$y(t_1) = 2x^2(t_1)$

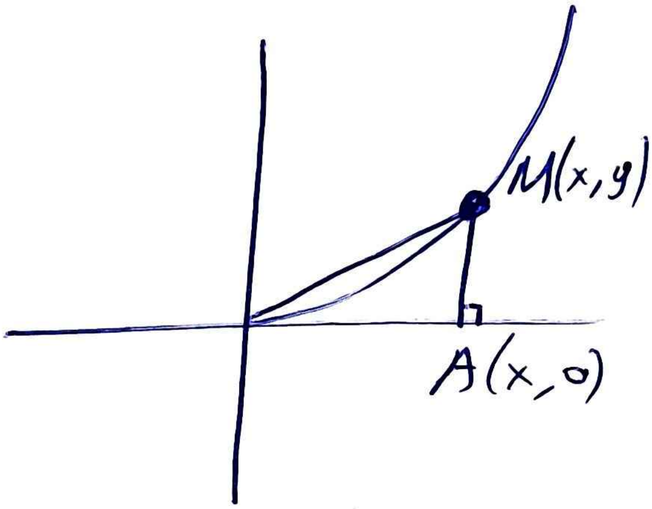
$y(t_1) = 2 \cdot 1^2$

$y(t_1) = 2$.

Ⓑ Να βρει το ρυθμο μεταβολη του εμβαδου του τριγωνου $\overset{\Delta}{MAO}$ οπου O αρχη των αξονων



M τυχαιο σημειο της C_f και A η προβολη του M πανω στον $x'x$ τη χρονικη στιγμη t_1.



$$E = \frac{x \cdot y}{2}$$

$$E(t) = \frac{1}{2} x(t) y(t)$$

$$E'(t) = \frac{1}{2} \cdot \left[x'(t) y(t) + x(t) y'(t)\right]$$

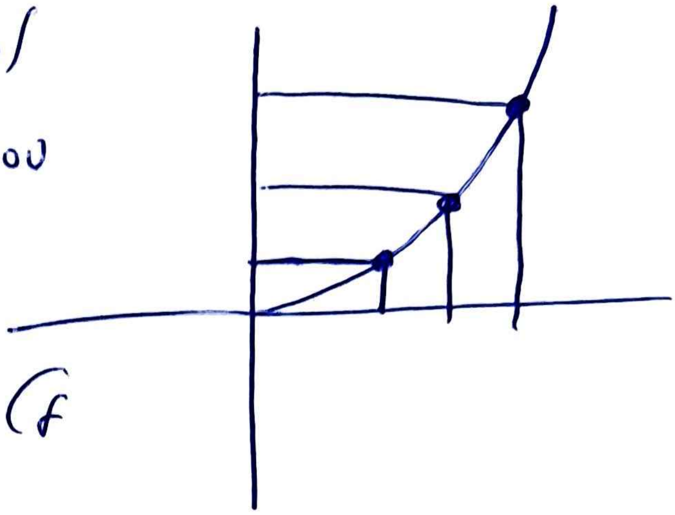
$$\underline{t = t_1}$$

$$E'(t_1) = \frac{1}{2} \left(x'(t_1) y(t_1) + x(t_1) y'(t_1) \right)$$

$$E'(t_1) = \frac{1}{2} \left[2 \cdot 2 + 1 \cdot 8 \right]$$

$$\boxed{E'(t_1) = 6.}$$

⑧ Να βρεθει ο ρυθμος μεταβολης του ορθογωνιου ΜΑΟΒ οπου



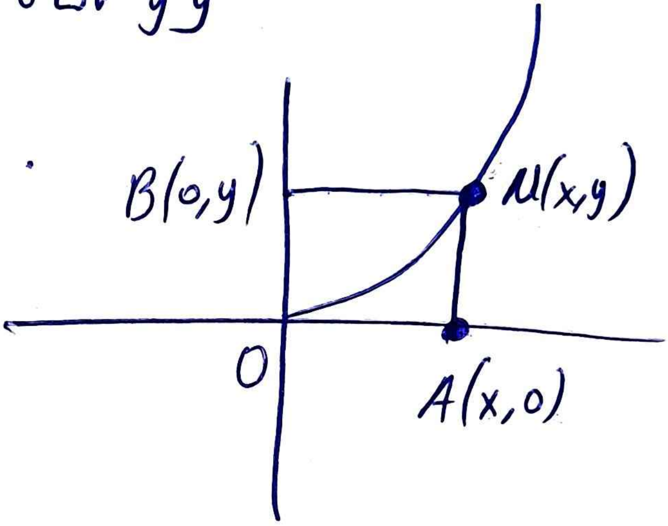
Μ τυχαιο σημειο της C_f

Ο αρχη αξονων

Α προβολη του Μ στον x'x

Β προβολη του Μ στον y'y

τη χρονικη στιγμη t_1.



$$\Pi = 2x + 2y$$

$$\Pi(t) = 2x(t) + 2y(t)$$

$$\Pi'(t) = 2x'(t) + 2y'(t)$$

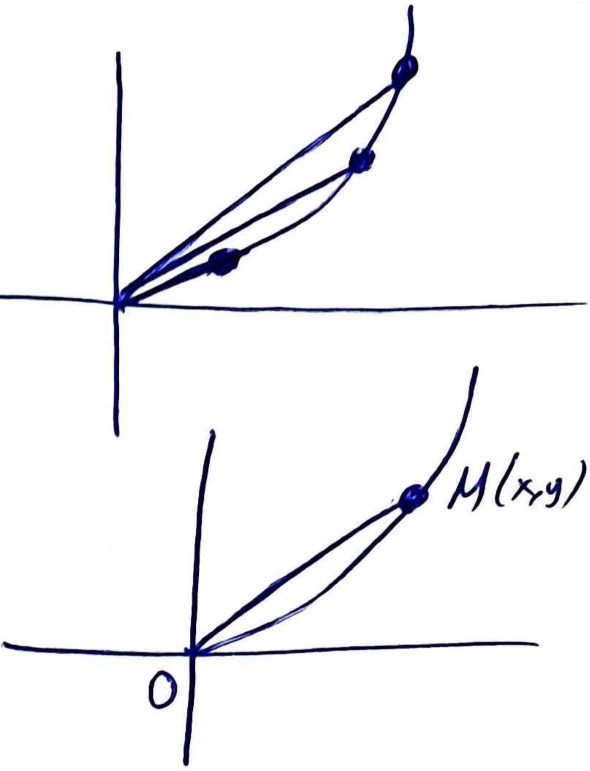
$\underline{t = t_1}$

$$\Pi'(t_1) = 2 \cdot x'(t_1) + 2y'(t_1)$$

$$\Pi'(t_1) = 2 \cdot 2 + 2 \cdot 8$$

$$\boxed{\Pi'(t_1) = 20}$$

(δ). Να Βρεθη ο ρυθμος μεταβολης της αποστασης του τυχαιου σημειου Μ της Cf απο την αρχη των αξονων τη χρονικη στιγμη t_1.



$$d_{OM} = \sqrt{(x-0)^2+(y-0)^2}$$

$$d = \sqrt{x^2+y^2}$$

$$d(t) = \sqrt{x^2(t)+y^2(t)}$$

$$d'(t) = \frac{1}{2\sqrt{x^2(t)+y^2(t)}} \cdot \left(x^2(t)+y^2(t)\right)'$$

$$d'(t) = \frac{1}{2\sqrt{x^2(t)+y^2(t)}} \cdot \left[2x(t)x'(t)+2y(t)y'(t)\right]$$

$\underline{t=t_1}$

$$d'(t_1) = \frac{1}{2\sqrt{x^2(t_1)+y^2(t_1)}} \cdot \left[2x(t_1)x'(t_1)+2y(t_1)y'(t_1)\right]$$

$$d'(t_1) = \frac{1}{2\sqrt{1+4}} \cdot \left[2\cdot 1\cdot 2+2\cdot 2\cdot 8\right] = \frac{36}{2\sqrt{5}} = \frac{18}{\sqrt{5}}.$$

Β' τρόπος

$d(t) = \sqrt{x^2(t)+y^2(t)}$

$d^2(t) = x^2(t) + y^2(t)$

$2d(t)\, d'(t) = 2x(t)\, x'(t) + 2y(t)\, y'(t)$

$\underline{t=t_1}$

$2d(t_1)\, d'(t_1) = 2x(t_1)\, x'(t_1) + 2y(t_1)\, y'(t_1)$

$2\sqrt{5}\; d'(t_1) = 2\cdot 1\cdot 2 + 2\cdot 2\cdot 8$

$2\sqrt{5}\, d'(t_1) = 36$

$d'(t_1) = \frac{36}{2\sqrt{5}}$

$d'(t_1) = \frac{18}{\sqrt{5}}$.

$d(t) = \sqrt{x^2(t)+y^2(t)}$

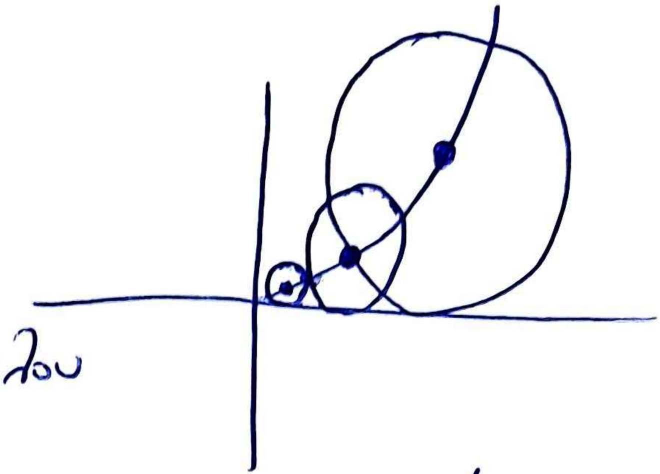
$\underline{t=t_1}$

$d(t_1) = \sqrt{x^2(t_1)+y^2(t_1)}$

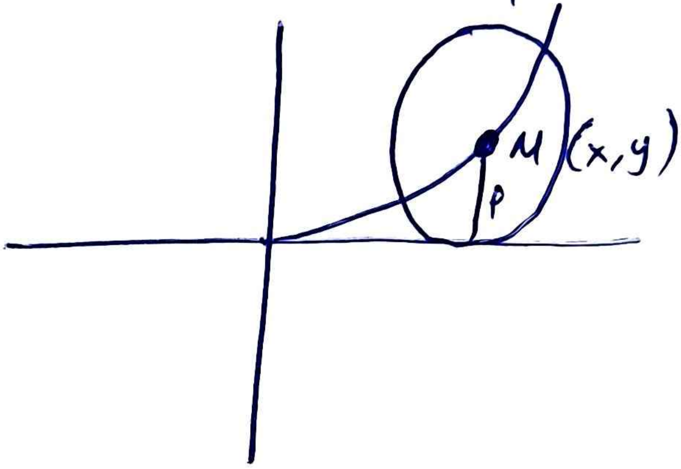
$d(t_1) = \sqrt{1+4}$

$d(t_1) = \sqrt{5}$.

(ε) . Να βρεθη ο ρυθμος μεταβολης του εμβαδου του κυκλου



με κεντρο το τυχαιο σημειο M της C_f που εφαπτεται στον x'x τη χρονικη στιγμη t_1.



$E=\pi\rho^2$

$E=\pi\cdot y^2$

$E(t)=\pi y^2(t)$

$E'(t)=2\pi y(t)\, y'(t)$

$\underline{t=t_1}$

$E'(t_1)=2\pi\, y(t_1)\, y'(t_1)$

$E'(t_1)=2\pi\cdot 2\cdot\ 8$

$\boxed{E'(t_1)=32\pi}$

Θεμα

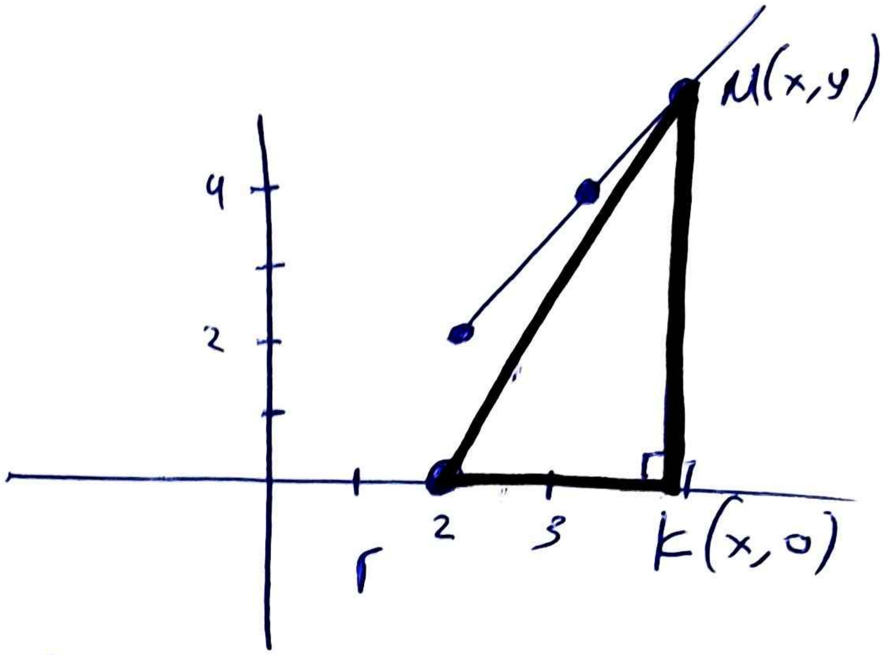
Εστω ευθεια $ε: y=2x-2$, $x>2$.

Ενα υλικο σημειο Μ κινειται κατα μηκος της ε.

Να βρεθει ο ρυθμος μεταβολης του εμβαδου του τριγωνου $\overset{\Delta}{ΜΚΓ}$ οπου Μ τυχαιο σημειο της ε, Γ(2,0) και Κ η προβολη του Μ πανω στον x'x τη χρονικη στιγμη t_0 κατα την οποιο το Μ διερχεται απο το σημειο Β(3,4).

Δινεται οτι ο ρυθμος μεταβολης της τετμημενης του Μ ειναι 2.

x	2	3
y	2	4



$$E = \frac{B \cdot U}{2}$$

$$E = \frac{(x-2)y}{2}$$

$$E(t) = \frac{1}{2}(x(t)-2)\,y(t)$$

$$E'(t) = \frac{1}{2}\left[x'(t)\,y(t) + (x(t)-2)\,y'(t)\right]$$

$$\frac{t = t_0}{E'(t_0) = \frac{1}{2}\left[x'(t_0)\; g(t_0) + (x(t_0)-2)\, g'(t_0)\right]}$$

$$E'(t_0) = \frac{1}{2}\left[2 \cdot 4 + (3-2) \cdot 4\right]$$

$$E'(t_0) = \frac{1}{2}\left[8 + 4\right] \qquad E'(t_0) = 6$$

$$\boxed{y = 2x - 2}$$

$$y(t) = 2x(t) - 2$$

$$y'(t) = 2x'(t) \qquad y'(t_0) = 4$$

$$y'(t_0) = 2x'(t_0)$$

$$y'(t_0) = 2 \cdot 2$$

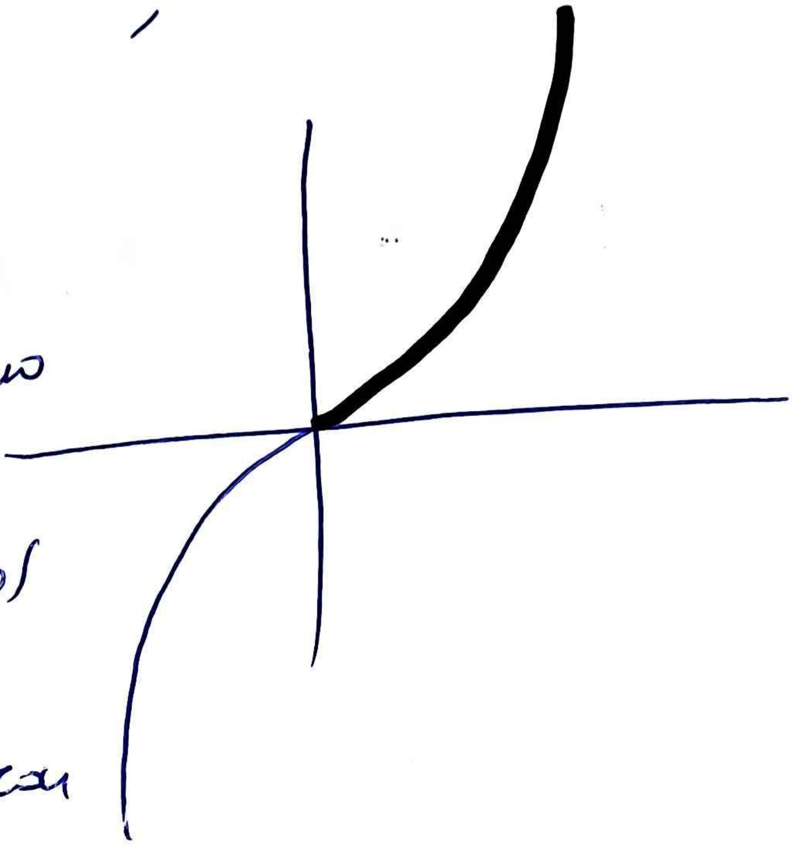
$$x(t_0) = 3$$

$$x'(t) = 2$$

$$y(t_0) = 4$$

Θεμα

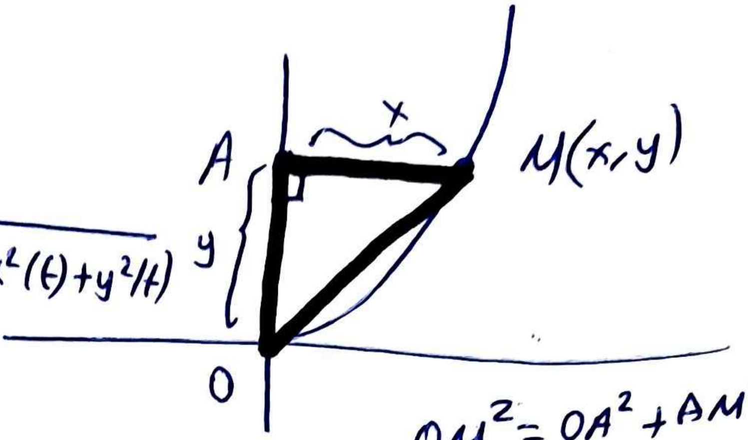
$f(x) = x^3 \quad , \quad x > 0$



Εστω t_1 η χρονικη στιγμη που το Μ διερχεται απο σημειο με τεταγμενη 8. Δινεται οτι ο ρυθμος μεταβολης της τετμημενης ελαττωνεται κατα 2.

Να βρεθει ο ρυθμος μεταβολης της περιμετρου του τριγωνου $O\hat{A}M$ οπου Α η προβολη του Μ πανω στον $y'y$ τη χρονικη στιγμη t_1.

$$\Pi = x + y + \sqrt{x^2+y^2}$$



$$\Pi(t) = x(t) + y(t) + \sqrt{x^2(t)+y^2(t)}$$

$OM^2 = OA^2 + AM^2$

$$\Pi'(t) = x'(t) + y'(t) + \frac{2x(t)x'(t) + 2y(t)y'(t)}{2\sqrt{x^2(t)+y^2(t)}}$$

$OM = \sqrt{x^2+y^2}$

$\underline{t = t_1}$

$x'(t_1) = -2$

$y(t_1) = 8$

$$\Pi'(t_1) = x'(t_1) + y'(t_1) + \frac{x(t_1)x'(t_1) + y(t_1)y'(t_1)}{\sqrt{x^2(t_1)+y^2(t_1)}}$$

$$\Pi'(t_1) = -2 \quad -24 \quad + \frac{2(-2) + 8\cdot(-24)}{\sqrt{4+16}}$$

$$\Pi'(t_1) = -26 + \frac{-4-192}{\sqrt{20}}$$

$y(t) = x^3(t)$

$y'(t) = 3x^2(t)x'(t)$

$\underline{t=t_1}$

$y'(t_1) = 3x^2(t_1)\,x'(t_1)$

$y'(t_1) = 12\cdot(-2) = -24$

$y'(t_1) = -24$

Αφού $y(t_1) = 8$

$8 = x^3(t_1)$ $x(t_1) = 2$

$f(x) = x^3$

$y = x^3$

$y(t) = x^3(t)$

$\underline{t=t_1}$

$y(t_1) = x^3(t_1)$

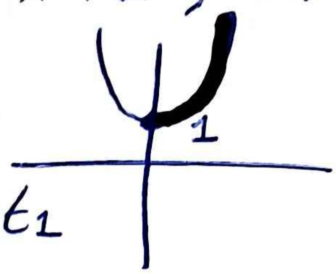
$$ת'(t_1) = -26 - \frac{196}{2\sqrt{5}}$$

$$\boxed{ת'(t_1) = -26 - \frac{98}{\sqrt{5}}.}$$

Εργασία ρυθμου μεταβολης

Δινεται η συναρτηση $f(x) = x^2+1$, $x>0$.

Ενα υλικο σημειο Μ κινειται κατα μηκος της C_f. Τη χρονικη στιγμη t_1 το Μ διερχεται απο το σημειο $A(1, f(1))$ και η τετμημενη αυξανεται κατα 1.



Να βρεθει ο ρυθμος μεταβολης τη χρονικη στιγμη t_1.

1. Της τεταγμενης του Μ

2. Του εμβαδου του τριγωνου $\overset{\triangle}{MOB}$ οπου Β η προβολη του Μ στον $x'x$.

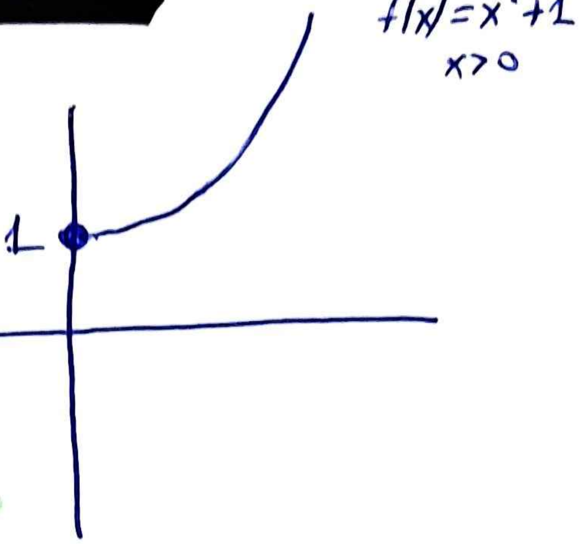
3. Του εμβαδου του τριγωνου $\overset{\triangle}{MO\Gamma}$ οπου $\Gamma(0,3)$

4. Της περιμετρου του ορθογωνιου ΜΒΟΔ οπου Δ η προβολη του Μ στον $y'y$.

5. Της αποστασης ΜΓ

6. Του μηκους κυκλου με κεντρο το Μ που εφαπτεται στον $y'y$.

Λυση Εργασιας



Αφου τη χρονικη στιγμη t_1 το Μ διερχεται απο το σημειο $A(1,2)$

$\Rightarrow$ $x(t_1)=1$ $y(t_1)=2$

$x'(t_1)=1$.

1. Ψαχνω το $y'(t_1)$

$f(x)=x^2+1$

$y=x^2+1$

$y(t)=x^2(t)+1$

$y'(t)=2x(t)\,x'(t)$.

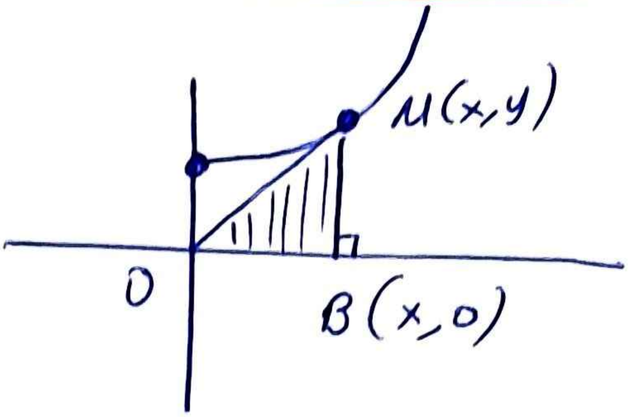
$\underline{t=t_1}$

$y'(t_1)=2x(t_1)\,x'(t_1)$

$y'(t_1)=2\cdot1\cdot1$

$y'(t_1)=2$

2. $E = \frac{B \cdot U}{2}$



$$E = \frac{x \cdot y}{2}$$

$$E(t) = \frac{1}{2}\ x(t)y(t)$$

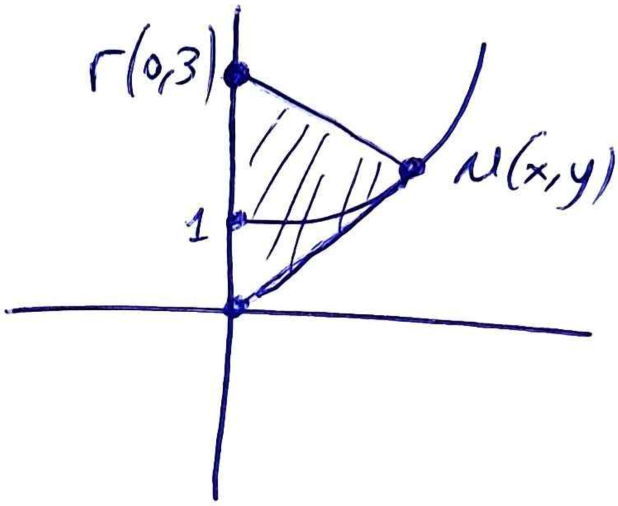
$$E'(t) = \frac{1}{2}\left(x'(t)\ y(t) + x(t)\ y'(t) \right)$$

$$\xrightarrow{t=t_1}$$

$$E'(t_1) = \frac{1}{2}\left(x'(t_1)y(t_1) + x(t_1)\ y'(t_1) \right)$$

$$E'(t_1) = \frac{1}{2}\left(1 \cdot 2 + 1 \cdot 2 \right)$$

$$E'(t_1) = 2$$



3. $E = \frac{B \cdot U}{2}$

$$E = \frac{3 \cdot x}{2}$$

$$E(t) = \frac{3}{2}\ x(t)$$

$$E'(t) = \frac{3}{2}\ x'(t)$$

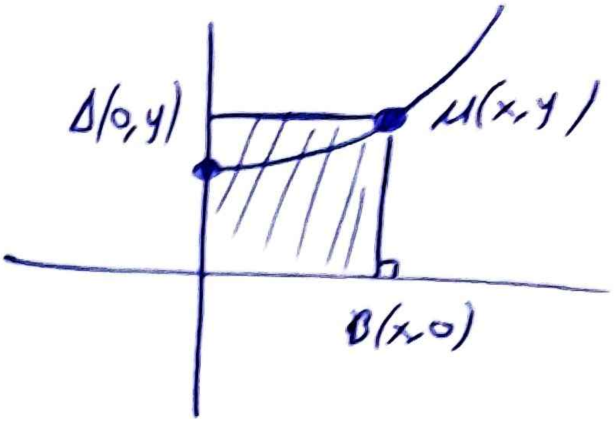
$$\xrightarrow{t=t_1}$$

$$E'(t_1) = \frac{3}{2}\ x'(t_1)$$

$$E'(t_1) = \frac{3}{2}\ 1$$

$$E'(t_1) = \frac{3}{2}$$

4. $\Pi = 2x+2y$



$\Pi(t) = 2x(t) + 2y(t)$

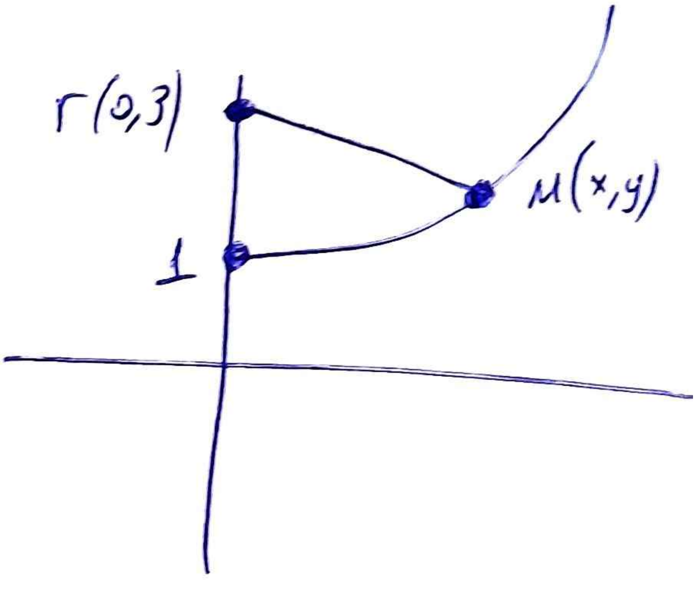
$\Pi'(t) = 2x'(t) + 2y'(t)$

$\underline{t=t_1}$

$\Pi'(t_1) = 2x'(t_1) + 2y'(t_1)$

$\Pi'(t_1) = 2\cdot 1 + 2\cdot 2$

$\Pi'(t_1) = 6$



5. $d = \sqrt{(x-0)^2+(y-3)^2}$

$d = \sqrt{x^2+(y-3)^2}$

$d(t) = \sqrt{x^2(t)+(y(t)-3)^2}$

$$d'(t) = \frac{2x(t)x'(t)+2(y(t)-3)y'(t)}{2\sqrt{x^2(t)+(y(t)-3)^2}}$$

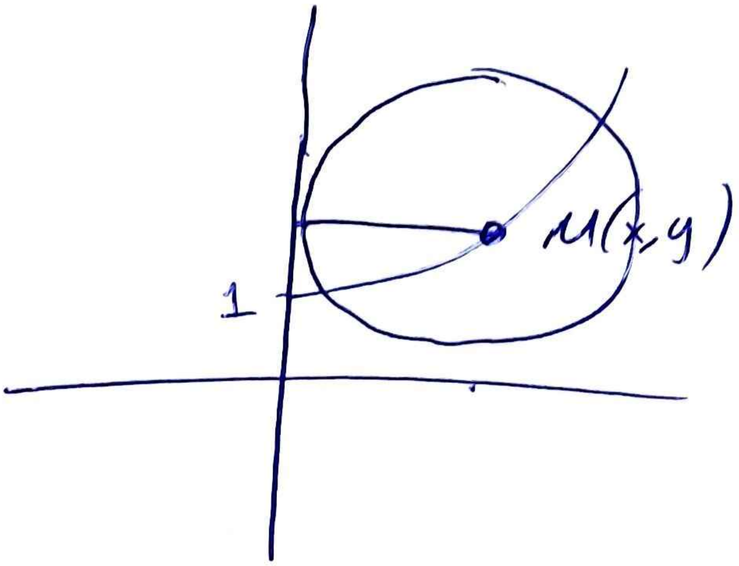
$\underline{t=t_1}$

$$d'(t_1) = \frac{2x(t_1)x'(t_1)+2(y(t_1)-3)y'(t_1)}{2\sqrt{x^2(t_1)+(y(t_1)-3)^2}}$$

$$d'(t_1) = \frac{2\cdot 1\cdot 1 + 2(2-3)2}{\sqrt{1^2+(2-3)^2}}$$

$$d'(t_1) = \frac{2 \; -4}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

$d'(t_1) = -\sqrt{2}$.



6. $E = \pi \rho^2$

$\epsilon = \pi x^2$

$\epsilon(t) = \pi x^2(t)$

$\epsilon'(t) = 2\pi x(t) x'(t)$

$\underline{t = t_1}$

$\epsilon'(t_1) = 2\pi x(t_1) \; x'(t_1)$

$\epsilon'(t_1) = 2\pi \cdot 1 \cdot 1$

$\epsilon'(t_1) = 2\pi$

Θεμα

Εστω υλικο σημειο Μ κινειται κατα μηκος της C_f οπου $f(x) = x^2, x>0$ και t_1 η χρονικη στιγμη που το Μ διερχεται απο σημειο με τεταγμενη 2.

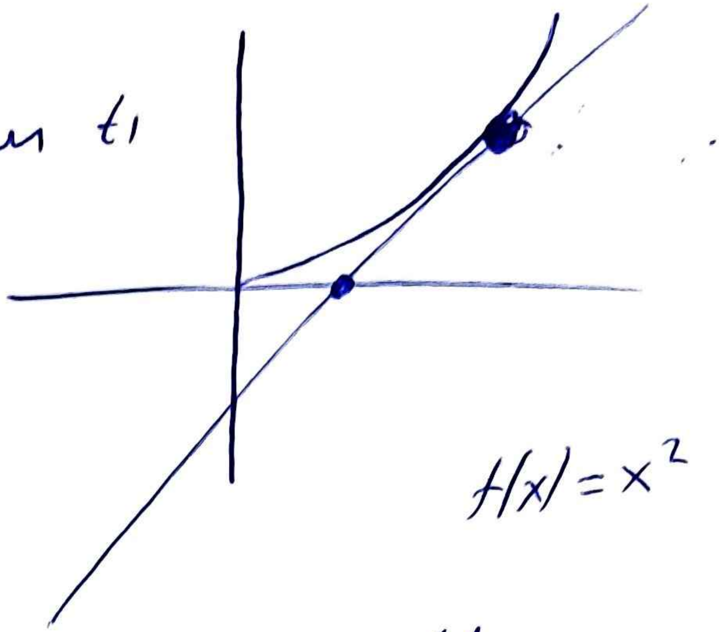
1. Να βρεθει ο ρυθμος μεταβολης της τεταγμενης, του σημειου τομης της εφαπτομενης της C_f στο Μ με τον $x'x$ τη χρονικη στιγμη t_1 εαν ειναι γνωστο οτι

$x'(t_1) = -1$.

1. Τη χρονικη στιγμη t_1

το $x(t_1)=2$

αρα $y(t_1)=4$



Εστω $M(\alpha, f(\alpha))$

Η τυχαια εφαπτομενη στο M.

$$y - f(a) = f'(a)(x-a)$$

Τεμνει τον x'x αν $y=0$.

$$0 - f(a) = f'(a)(x-a)$$

$$-f(a) = f'(a)x - af'(a)$$

$$af'(a) - f(a) = f'(a)x$$

$$\boxed{x = \frac{af'(a) - f(a)}{f'(a)}}$$

αυτη η τυχαια τετμημενη του σημειου τομης της εφαπτομενης με τον x'x.

$$x = \frac{\alpha \cdot 2\alpha - \alpha^2}{2\alpha} = \frac{\alpha^2}{2\alpha} = \frac{\alpha}{2}$$

Άρα $x = \frac{a}{2}$

$$x(t) = \frac{1}{2} a(t)$$

$$x'(t) = \frac{1}{2} a'(t)$$

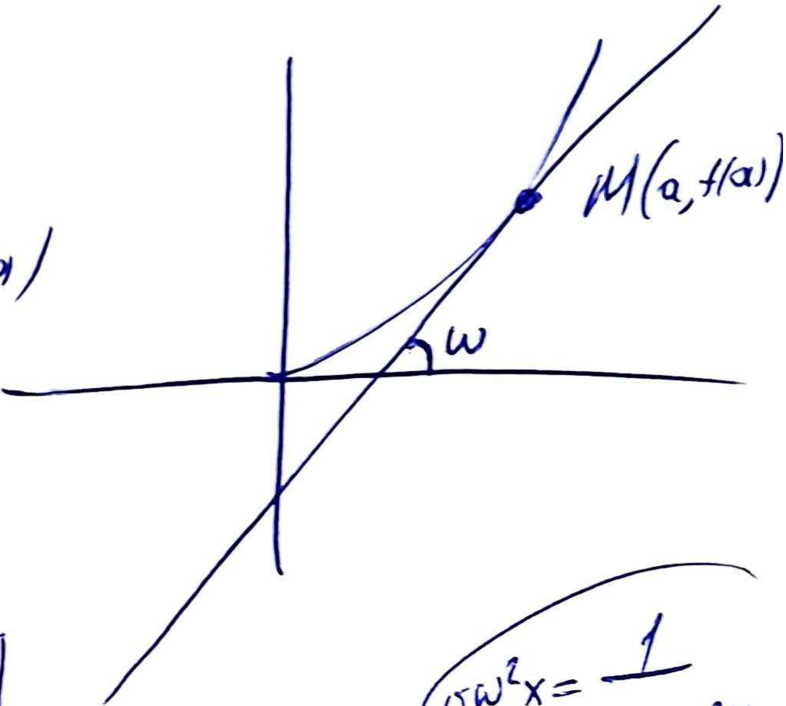
$$\xrightarrow{t=t_1}$$

$$x'(t_1) = \frac{1}{2} (-1)$$

$$x'(t_1) = -\frac{1}{2}$$

ο ζητούμενος ρυθμός μεταβολής.

2. Να βρεθει ο ρυθμος μεταβολης της γωνιας που σχηματιζει η εφαπτομενη της Cf στο M με τον x'x τη χρονικη στιγμη t1.



$$\varepsilon: \; y - f(a) = f'(a)(x-a)$$

Γνωριζω $\text{εφ}\hat{\omega} = f'(a)$

$$\text{εφ}\hat{\omega} = 2\alpha$$

$$\boxed{\text{εφ}\hat{\omega}(t) = 2\alpha(t)}$$

$$\text{συν}^2 x = \frac{1}{1+\text{εφ}^2 x}$$

$$\frac{\omega'(t)}{\text{συν}^2\omega(t)} = 2\alpha'(t)$$

$$\omega'(t) = 2\alpha'(t)\,\text{συν}^2\omega(t)$$

$$\omega'(t) = 2\alpha'(t)\,\frac{1}{1+\text{εφ}^2\omega(t)}$$

$$\boxed{\omega'(t) = \frac{2\alpha'(t)}{1+4\alpha^2(t)}}$$

$$\underline{t = t_1}$$

$$\omega'(t_1) = \frac{2a'(t_1)}{1+4a^2(t_1)}$$

$$\omega'(t_1) = \frac{2(-1)}{1+4\cdot 2}$$

$$\boxed{\omega'(t_1) = \frac{-2}{9}}.$$

Θεμα Γ (2024)

Εστω $f(x) = \begin{cases} -2x+4+e^{\lambda}, & 0 \le x < 2 \\ -x^2+4x-3+\lambda, & x \ge 2 \end{cases}$

Συνεχη!.

Γ1 Νδο $\lambda = 0$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-2x+4+e^{\lambda}) = e^{\lambda}$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x^2+4x-3+\lambda) = 1+\lambda$$

$$e^{\lambda} = 1+\lambda .$$

$$e^{\lambda} - 1 - \lambda = 0 \longrightarrow \varphi(\lambda) = 0$$

$\varphi(\lambda) = \varphi(0)$

φ 1-1

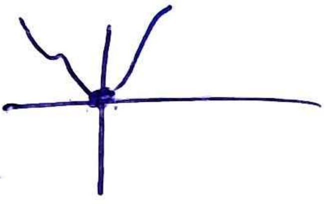
$\underline{\underline{\lambda = 0}}$

$$\boxed{\varphi(x) = e^x - 1 - x}$$

$\varphi'(x) = e^x - 1$

$\rightarrow \varphi'(x) = 0 \Rightarrow e^x - 1 = 0 \Rightarrow e^x = 1 \Rightarrow \underline{\underline{x = 0}}$

x		0	
φ'	−	0	+
φ	↘		↗



Γ2

$$f(x) = \begin{cases} -2x+5, & 0 \leq x \leq 2 \\ -x^2+4x-3, & x > 2 \end{cases}$$

$0 \leq x < 2$

$f_1(x) = -2x+5$

$f_1'(x) = -2 < 0$

$f_1 \downarrow$

$x > 2$

$f_2(x) = -x^2+4x-3$

$f_2'(x) = -2x+4$

$f_2'(x) = -2\overset{\oplus}{(x-2)}$

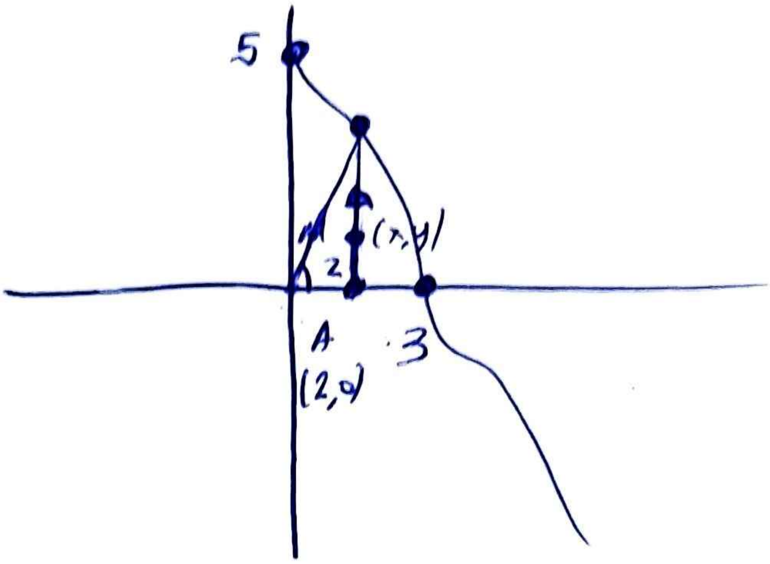
$f_2'(x) < 0$

$f_2 \downarrow$

$f \downarrow$ και δεν

έχει ακρότατα

Γ4



$f(x)=0$

$0 \le x < 2$

$f(x)=0$

$-2x+5=0$

$5=2x$

$x=\frac{5}{2}$

Απορ/ται

$x > 2$

$f(x)=0$

$-x^2+4x-3=0$

$x^2-4x+3=0$

$\cancel{x=1}$ $\boxed{x=3}$

Το Μ κινειται με σταθερη ταχυτητα 0,5 m/s

$\hookrightarrow y'(t)=0,5$

$$\varepsilon\varphi\hat{\omega} = \frac{y}{2}$$

$$\boxed{\varepsilon\varphi\hat{\omega}(t) = \frac{1}{2}\, y(t)}$$

$$\frac{\omega'(t)}{\sigma w^2\omega(t)} = \frac{1}{2}\, y'(t)$$

$$\omega'(t) = \frac{1}{2}\, y'(t) \quad \sigma w^2\omega(t)$$

$$\omega'(t) = \frac{1}{2}\, y'(t) \quad \frac{1}{1+\ \varepsilon\varphi^2\omega(t)}$$

$$\omega'(t) = \frac{1}{2}\, y'(t) \cdot \frac{1}{1+\left(\frac{1}{2}\, y(t)\right)^2}$$

$$\underline{t = t_1}$$

$$\omega'(t_1) = \frac{1}{2}\, y'(t_1) \cdot \frac{1}{1+\frac{1}{4}\ y^2(t_1)}$$

$$\omega'(t_1) = \frac{1}{2}\,\frac{1}{2} \cdot \frac{1}{1+\frac{1}{4}\cdot 1} = \frac{1}{4}\cdot\frac{1}{\frac{5}{4}} \boxed{= \frac{1}{5}}$$

Σελ 287

$f(2)=3$

$\boxed{f'(2)=1.}$ κρυβει οριο

$\downarrow$

$$\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=1$$

$$\boxed{\lim_{x\to 2}\frac{f(x)-3}{x-2}=1.}$$

Θετω $g(x)=\frac{f(x)-3}{x-2}$ $\Leftrightarrow$ $\boxed{f(x)=g(x)(x-2)+3}$

ⓐ $$\lim_{x\to 2}\frac{xf(x)-6}{x-2}=\lim_{x\to 2}\frac{x\left(g(x)(x-2)+3\right)-6}{x-2}$$

$$=\lim_{x\to 2}\frac{xg(x)(x-2)+3x-6}{x-2}=\lim_{x\to 2}\frac{x\cancel{(x-2)}g(x)}{\cancel{x-2}}+\frac{3\cancel{(x-2)}}{\cancel{x-2}}$$

$$=2\cdot 1+3=5$$

Ⓑ $\lim\limits_{x \to 2} \dfrac{2f(x) - 3x}{x^2 - 2x} = \lim\limits_{x \to 2} \dfrac{2\left(g(x)(x-2)+3\right) - 3x}{x^2 - 2x}$

$= \lim\limits_{x \to 2} \dfrac{2g(x)(x-2) + 6 - 3x}{x^2 - 2x} =$

$= \lim\limits_{x \to 2} \dfrac{2g(x)\cancel{(x-2)} - 3\cancel{(x-2)}}{x\cancel{(x-2)}} = \dfrac{2 \cdot 1 - 3}{2} = -\dfrac{1}{2}$

19. Εστω $f(2)=0$ και $f'(2)=-2$ κρυβει οριο.

Βρε $\lim\limits_{x\to 1} \dfrac{f(3x-1)}{x-1}$

$$\lim_{x\to 1} \frac{f(3x-1)}{x-1} \overset{\substack{3x-1=t \\ 3x=t+1 \\ x=\frac{t+1}{3} \\ x\to 1 \\ t\to 2}}{=\!=\!=} \lim_{t\to 2} \frac{f(t)}{\frac{t+1}{3}-1} =$$

$$= \lim_{t\to 3} \frac{3f(t)}{t+1-3} = \lim_{t\to 3} \frac{3f(t)}{t-2} =$$

$$= \lim_{x\to 3} \frac{3f(x)}{x-2} = 3\cdot(-2) = -6$$

$$f'(2) = -2 \iff \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = -2$$

$$\iff \lim_{x\to 2} \frac{f(x)-0}{x-2} = -2 \iff \lim_{x\to 2} \frac{f(x)}{x-2} = -2$$

f συνεχής στο x_0.

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Θέτω $x = x_0 + h$

$x \to x_0$

$h \to 0$

$$\lim_{h\to 0} f(x_0+h) = f(x_0)$$

εναλλακτικός ορισμός της συνέχειας.

f παραγ/μη στο x_0

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \xrightarrow[\substack{x=h+x_0\\ x\to x_0\\ h\to 0}]{x-x_0=h} \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}.$$

εναλλακτικός ορισμός της παραγώγου στο x_0.

20. (α) νδο $\lim_{h\to 0} \frac{f(2+3h)-f(2)}{h} = 3f'(2)$

$$\lim_{h\to 0} \frac{f(2+3h)-f(2)}{h} \xrightarrow[3h=x-2]{2+3h=x} \lim_{x\to 2} \frac{f(x)-f(2)}{\frac{x-2}{3}}$$

$3h = x-2$

$h = \frac{x-2}{3}$

$h\to 0$

$x\to 2$

$$= \lim_{x\to 2} 3\,\frac{f(x)-f(2)}{x-2} = 3f'(2).$$

(B). $\lim_{h\to 0} \frac{f(2+h)-f(2-h)}{h} = 2f'(2)$

$$\lim_{h\to 0} \frac{f(2+h)-f(2-h)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)+f(2)-f(2-h)}{h}$$

$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} - \frac{f(2-h)-f(2)}{h} \quad ⊛$$

$$\bullet \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \xlongequal[\substack{h=x-2\\ h\to 0\\ x\to 2}]{2+h=x} \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=f'(2)$$

$$\bullet \lim_{h\to 0}\frac{f(2-h)-f(2)}{h} \xlongequal[\substack{h=2-x\\ h\to 0\\ x\to 2}]{2-h=x} \lim_{x\to 2}\frac{f(x)-f(2)}{2-x}$$

$$=-\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=-f'(2)$$

$$\xlongequal{\textcircled{*}} f'(2)-\left(-f'(2)\right)=2f'(2)$$

$$\textcircled{\gamma}\ \lim_{h\to +\infty}\frac{h\cancel{(h^2+1)}}{\cancel{h^2+1}}\left(f\left(2+\frac{1}{h}\right)-f(2)\right)=f'(2)$$

$$\to \lim_{h\to +\infty} h\cdot\left(f\left(2+\frac{1}{h}\right)-f(2)\right) \xlongequal[\substack{h\to +\infty\\ x\to 2\\ \frac{1}{h}=x-2\ \Rightarrow h=\frac{1}{x-2}}]{2+\frac{1}{h}=x}$$

$$\lim_{x\to 2} \frac{1}{x-2}\left(f(x)-f(2)\right) = f'(2).$$

Επομαω Μαθημα

Σελ 375

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Σελ. 286

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⑯

⑱

26.

$$f(x)= e^{x}+x^{2}-\frac{2}{3}x+1.$$

$$x'(t)=3y'(t) \qquad x'(t)>0.$$

$$y=e^{x}+x^{2}-\frac{2}{3}x+1$$

$$y(t)=e^{x(t)}+x^{2}(t)-\frac{2}{3}x(t)+1$$

$$y'(t)=e^{x(t)}x'(t)+2x(t)x'(t)-\frac{2}{3}x'(t)$$

$$\frac{1}{3}\cancel{x'(t)}=e^{x(t)}\cancel{x'(t)}+2x(t)\cancel{x'(t)}-\frac{2}{3}\cancel{x'(t)}$$

$$\frac{1}{3}=e^{x(t)}+2x(t)-\frac{2}{3}$$

$$1=e^{x(t)}+2x(t)$$

$$e^{x(t)}+2x(t)-1=0$$

$\varphi(x) = e^x + 2x - 1$

$\varphi \nearrow$

$\varphi(x(t)) = \varphi(0)$

$\varphi : 1-1$

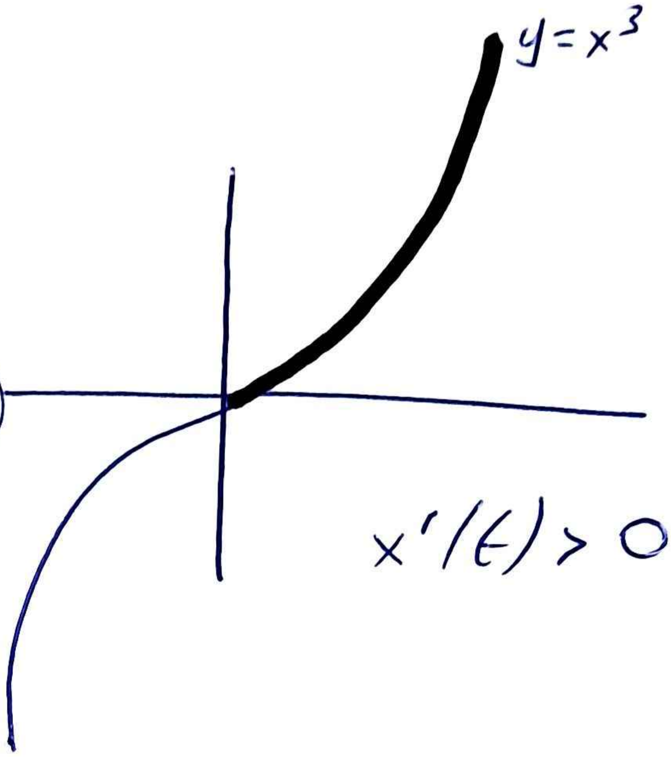
$\underline{\underline{x(t) = 0}}$

$\underline{\underline{A(0,2)}}$

Σελ 375

13.

$\boxed{y'(t) = x'(t)}$



$x'(t) > 0$

$y = x^3$

$y(t) = x^3(t)$

$y'(t) = 3x^2(t)\ x'(t)$

$\cancel{x'(t)} = 3x^2(t)\ \cancel{x'(t)}$

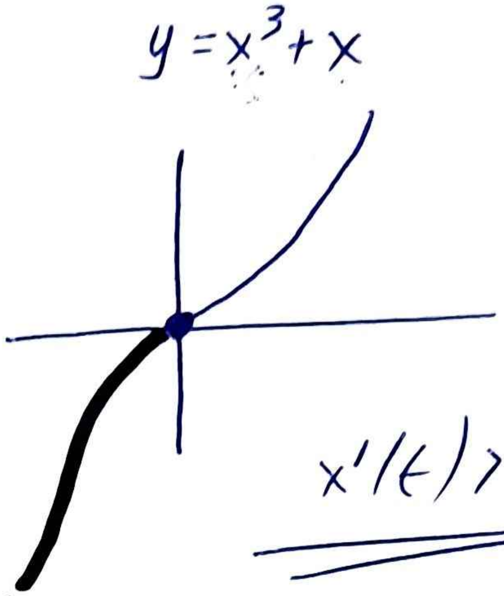
$1 = 3x^2(t)$

$x^2(t) = \frac{1}{3}$

$\boxed{x(t) = \frac{\sqrt{3}}{3}}$

14.

$$\boxed{y'(t)=4x'(t)}$$



$x'(t)>0$

$y = x^3+x$

$y(t) = x^3(t) + x(t)$

$y'(t) = 3x^2(t)\,x'(t) + x'(t)$

$4\cancel{x'(t)} = 3x^2(t)\,\cancel{x'(t)} + \cancel{x'(t)}$

$4 = 3x^2(t) + 1$

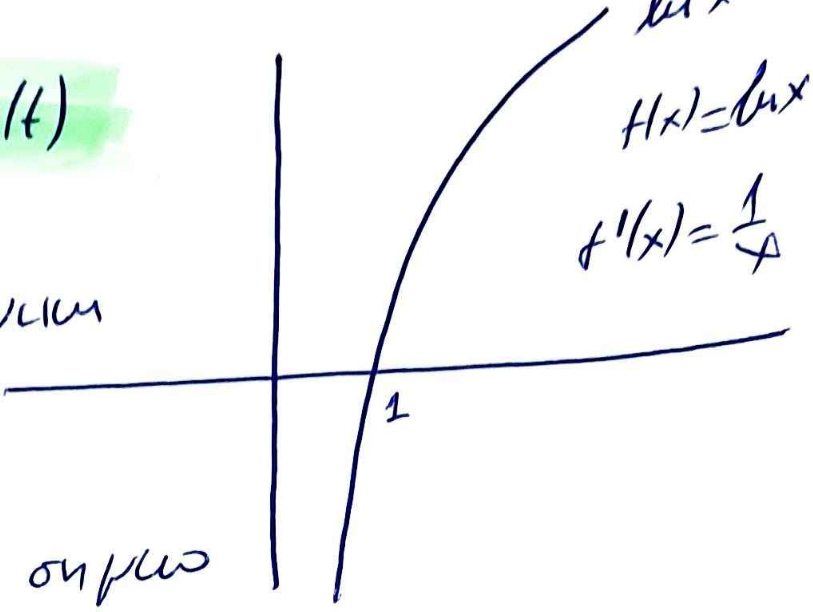
$3x^2(t) = 3$

$x^2(t) = 1$

$x(t) = -1$

$A(-1, -2)$.

22. $\alpha'(t) = 2\alpha(t)$



$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

Εστω t_1 η χρονικη στιγμη που το Μ διερχεται απο σημειο με τετμημενη e .

$x(t_1) = e$

$y(t_1) = 1$

$y - f(\alpha) = f'(\alpha)(x-\alpha)$

$y - \ln\alpha = \frac{1}{\alpha}(x-\alpha)$

$\underline{x'x \quad (y=0)}$

$0 - \ln\alpha = \frac{1}{\alpha}(x-\alpha)$

$-\alpha\ln\alpha = x - \alpha$

$x = \alpha - \alpha\ln\alpha$

$x = \alpha(1-\ln\alpha)$

$x(t) = \alpha(t)(1-\ln\alpha(t))$

$x'(t) = \alpha'(t)(1-\ln\alpha(t)) - \alpha(t)\frac{\alpha'(t)}{\alpha(t)}$

$$x'(t) = 2a(t)\left(1 - b_1 a(t)\right) - \cancel{a(t)}\,\frac{2a(t)}{\cancel{a(t)}}$$

$$\underleftarrow{t = t_1}$$

$$x'(t_1) = 2a(t_1)\left(1 - b_1 a(t_1)\right) - 2a(t_1)$$

$$x'(t_1) = 2e\cancel{(1 - b_1 e)} - 2e$$

$$x'(t_1) = -2e$$

Ⓑ $\varepsilon\varphi\hat{\omega} = f'(a)$

$$\varepsilon\varphi\hat{\omega} = \frac{1}{\alpha}$$

$$\varepsilon\varphi\hat{\omega}(t) = \frac{1}{a(t)}$$

$$\frac{\omega'(t)}{\sigma w^2 \omega(t)} = -\frac{1}{a^2(t)}\, a'(t)$$

$$\omega'(t) = -\frac{2a(t)}{a^2(t)}\, \sigma w^2 \omega(t)$$

$$\omega'(t) = -\frac{2}{a(t)}\, \frac{1}{1+\varepsilon\varphi^2\omega(t)}$$

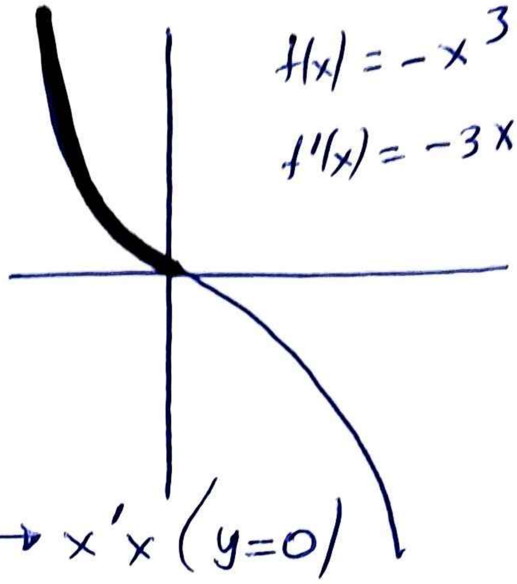
$$\omega'(t) = -\frac{2}{a(t)}\, \frac{1}{1+\frac{1}{a^2(t)}}$$

$\underline{t = t_1}$

$$\omega'(t_1) = -\frac{2}{e}\, \frac{1}{1+\frac{1}{e^2}} = -\frac{2}{\cancel{e}} \cdot \frac{e^{\cancel{2}}}{1+e^2}$$

$$\omega'(t_1) = \frac{-2e}{e^2+1}$$

21.



$f(x) = -x^3$

$f'(x) = -3x^2$

ⓐ $y - f(a) = f'(a)(x-a)$

$y - (-a^3) = -3a^2(x-a)$

$y + a^3 = -3a^2(x-a) \longrightarrow x'x\ (y=0)$

$a^3 = -3a^2(x-a)$

$\boxed{y = -x^3}$

$a = -3(x-a)$

$a = -3x + 3a$

$3x = 2a$

$x = \frac{2a}{3} \longrightarrow K\left(\frac{2a}{3}, 0\right)$.

Σημειο τομης εφαπτομενης με $x'x$.

$a'(t) = 2a(t)$

ⓑ Εστω t_1 η χρονικη στιγμη που το Μ εχει τετμημενη -6

$x(t_1) = -6$

$x = \frac{2a}{3} \Rightarrow x(t) = \frac{2}{3}a(t)$

$x'(t) = \frac{2}{3}a'(t) = \frac{2}{3}2a(t)$

Αρα $x'(t) = -\frac{4}{3} a(t)$

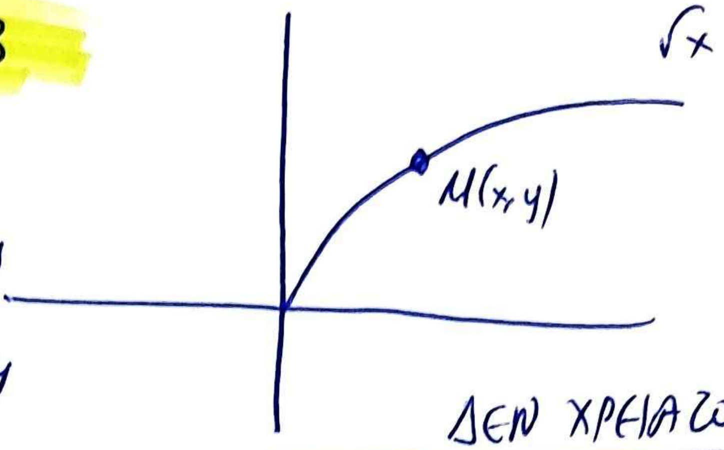
$t = t_1$

$$x'(t_1) = -\frac{4}{3} a(t_1)$$

$$x'(t_1) = -\frac{4}{3}(-6)$$

$$x'(t_1) = 8$$

17. $d'(t)=3$



Εστω t_1 η χρονικη στιγμη που το $x=4$

$\Rightarrow x(t_1)=4$

$y(t_1)=2$.

$$d=\sqrt{(x-0)^2+(y-0)^2}$$

$$d(t)=\sqrt{x^2(t)+y^2(t)}$$

$$d(t)=\sqrt{x^2(t)+\sqrt{x(t)}^2}$$

$$d(t)=\sqrt{x^2(t)+x(t)}$$

$$d'(t)=\frac{1}{2\sqrt{x^2(t)+x(t)}}\left(2x(t)x'(t)+x'(t)\right)$$

ΔΕΝ ΧΡΕΙΑΖΟΝΤΑΙ

$$y=\sqrt{x}$$

$$y(t)=\sqrt{x(t)}$$

$$y'(t)=\frac{x'(t)}{2\sqrt{x(t)}}$$

$t=t_1$

$$y'(t_1)=\frac{x'(t_1)}{2\sqrt{x(t_1)}}$$

$$y'(t_1)=\frac{x'(t_1)}{4}.$$

$\underline{t = t_1}$

$$d'(t_1) = \frac{2x(t_1)x'(t_1) + x'(t_1)}{2\sqrt{x^2(t_1) + x(t_1)}}$$

$$3 = \frac{2 \cdot 4\, x'(t_1) + x'(t_1)}{2\sqrt{20}}$$

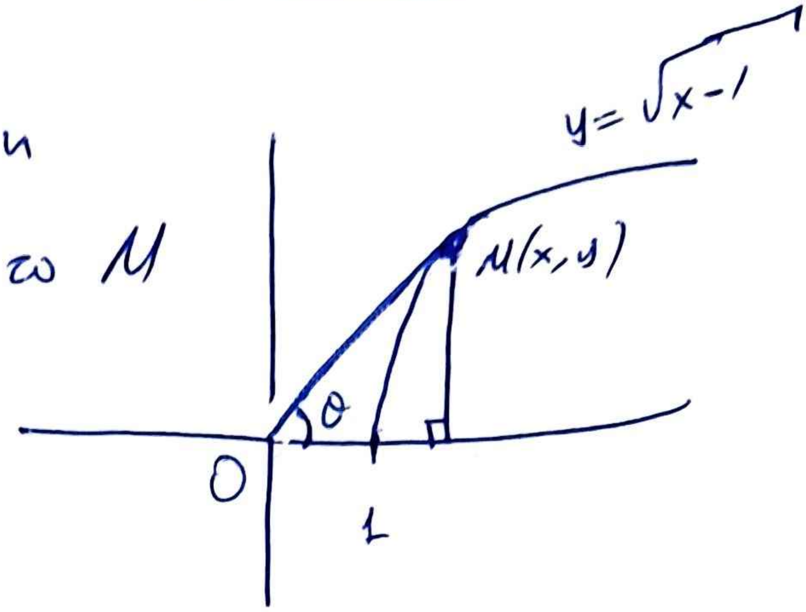
$$6\sqrt{20} = x'(t_1)\,(8 + 1)$$

$$6\sqrt{4}\sqrt{5} = 9\,x'(t_1)$$

$$\frac{12}{9}\sqrt{5} = x'(t_1)$$

$$\boxed{x'(t_1) = \frac{4\sqrt{5}}{3}}$$

18. Εστω t_1 η χρονικη στιγμη που το M διερχεται απο το σημειο $A(5,2)$



$\Rightarrow \quad x(t_1)=5$

$\Rightarrow \quad y(t_1)=2$

$x'(t_1)=-3$

$$\varepsilon\varphi\,\hat{\theta}=\frac{y}{x}$$

$$\varepsilon\varphi\,\hat{\theta}(t)=\frac{y(t)}{x(t)}$$

$$y=\sqrt{x-1}$$

$$y(t)=\sqrt{x(t)-1}$$

$$y'(t)=\frac{x'(t)}{2\sqrt{x(t)-1}}$$

$\underline{t=t_1}$

$$y'(t_1)=\frac{-3}{2\cdot 2}$$

$$y'(t_1)=-\frac{3}{4}$$

$$\frac{\theta'(t)}{\sigma\upsilon\nu^2\theta(t)}=\frac{y'(t)\,x(t)+y(t)\,x'(t)}{x^2(t)}$$

$$\theta'(t)=\frac{y'(t)\,x(t)+y(t)\,x'(t)}{x^2(t)}\quad\sigma\upsilon\nu^2\theta(t)$$

$$\theta'(t)=\frac{y'(t)\,x(t)+y(t)\,x'(t)}{x^2(t)}\cdot\frac{1}{1+\varepsilon\varphi^2\theta(t)}$$

$$\theta'(t) = \frac{y'(t)\,x(t) - y(t)\,x'(t)}{x^2(t)} \cdot \frac{1}{1+\left(\frac{y(t)}{x(t)}\right)^2}$$

$t = t_1$

$$\theta'(t_1) = \frac{y'(t_1)\,x(t_1) - y(t_1)\,x'(t_1)}{x^2(t_1)} \frac{1}{1+\left(\frac{y(t_1)}{x(t_1)}\right)^2}$$

$$\theta'(t_1) = \frac{-\frac{3}{4}\cdot 5 - 2(-3)}{25} \cdot \frac{1}{1+\left(\frac{2}{5}\right)^2}$$

$$\theta'(t_1) = \frac{-\frac{15}{4} + 6}{25} \cdot \frac{1}{1+\frac{4}{25}}$$

$$\theta'(t_1) = \frac{-\frac{9}{4}}{\cancel{25}} \cdot \frac{1}{\frac{29}{\cancel{25}}} = -\frac{9}{4\cdot 29}.$$

$$\theta'(t_1) = -\frac{9}{116}$$

16. Δινεται $f(0)=0$ και $f'(0)=1$

$$\downarrow$$

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=1$$

$$\boxed{\lim_{x\to 0}\frac{f(x)}{x}=1}$$

(α) $\lim\limits_{x\to 0}\dfrac{f(x)}{x}=1$.

(β). $\lim\limits_{x\to 0}\dfrac{f^2(x)-3f(x)}{x^2+x}=$

$$=\lim_{x\to 0}\frac{f(x)\,\frac{f(x)}{x}-3\,\frac{f(x)}{x}}{x+1}=$$

$$=\frac{0\cdot 1-3\cdot 1}{0+1}=-3.$$

(γ) $\lim\limits_{x\to 0}\dfrac{f(x)}{ημ2x}=\lim\limits_{x\to 0}\dfrac{\frac{f(x)}{x}}{2\,\frac{ημ2x}{2x}}=\dfrac{1}{2\cdot 1}=\dfrac{1}{2}$.

Σελ 286

15. Δίνεται $f(1)=2$ και $f'(1)=2$

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=2$$

$$\boxed{\lim_{x\to 1}\frac{f(x)-2}{x-1}=2}$$

Θέτω $g(x)=\dfrac{f(x)-2}{x-1}$

$g(x)(x-1)=f(x)-2$

$f(x)=g(x)(x-1)+2$

ⓐ $$\lim_{x\to 1}\frac{f(x)-2x^2}{x^2-x}=$$

$$=\lim_{x\to 1}\frac{g(x)(x-1)+2-2x^2}{x^2-x}=$$

$$=\lim_{x\to 1}\frac{g(x)\cancel{(x-1)}-2\cancel{(x-1)}(x+1)}{x\cancel{(x-1)}}$$

$$=\frac{2-2\cdot 2}{1}=-2$$

ⓑ $$\lim_{x\to 1}\frac{xf(x)-2}{x^2-1}=\lim_{x\to 1}\frac{x\left(g(x)(x-1)+2\right)-2}{x^2-1}$$

$$=\lim_{x\to 1}\frac{xg(x)(x-1)+2x-2}{(x-1)(x+1)}=$$

$$=\lim_{x\to 1}\frac{xg(x)\cancel{(x-1)}+2\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}=\frac{1\cdot 2+2}{2}=2$$

18.

(α) Νδο $\lim\limits_{x\to a}\dfrac{xf(x)-af(a)}{x-a}=f(a)+af'(a)$.

$$\to \underset{x\to a}{\ell}\frac{xf(x)-af(a)}{x-a}=\underset{x\to a}{\ell}\frac{xf(x)-xf(a)+xf(a)-af(a)}{x-a}$$

$$=\underset{x\to a}{\ell}\frac{x(f(x)-f(a))+f(a)(x-a)}{x-a}$$

$$=\underset{x\to a}{\ell}\,x\,\frac{f(x)-f(a)}{x-a}+f(a)\frac{\cancel{x-a}}{\cancel{x-a}}$$

$$=a\cdot f'(a)+f(a).$$

Αφού f παρ/μη στο α

$\underset{x\to a}{\ell}\dfrac{f(x)-f(a)}{x-a}$ υπάρχει $\ell\in\mathbb{R}$.

Ⓑ Νδο $\lim_{x\to a} \frac{af(x)-xf(a)}{x^2-ax} = f'(a) - \frac{f(a)}{a}$

$$\lim_{x\to a} \frac{af(x)-xf(a)}{x^2-ax} = \lim_{x\to a} \frac{af(x)-af(a)+af(a)-xf(a)}{x^2-ax}$$

$$= \lim_{x\to a} \frac{a(f(x)-f(a)) - f(a)(x-a)}{x(x-a)}$$

$$\lim_{x\to a} \frac{a}{x}\frac{f(x)-f(a)}{x-a} - \frac{f(a)}{x}\cancel{\frac{x-a}{x-a}}$$

$$= \frac{a}{a} f'(a) - \frac{f(a)}{a}$$

$$= f'(a) - \frac{1}{a} f(a).$$ ✓

Προσοχη

Εστω f συνεχης στο Δ.

1. Αν $f'(x) > 0$ σε καθε εσωτερικο x του Δ τοτε η f ↗

2. Αν $f'(x) < 0$ σε καθε εσωτερικο x του Δ τοτε η f ↘

Σελ 72

2. (β) $f(x) = \frac{1}{x} - \ln x$

Πεπ $x \neq 0$ και $x > 0$

$D_f = (0, +\infty)$.

$f'(x) = -\frac{1}{x^2} - \frac{1}{x} < 0$ για $x > 0$.

$f \downarrow$ στο $(0, +\infty)$.

(δ) $f(x) = 2x + συνx - 1$. $D_f = R$.

$f'(x) = 2 - ημx > 0$ $f \nearrow$

- $-1 \leq ημx \leq 1 \Rightarrow 2 \geq -ημx \geq -1$

$4 \geq 2 - ημx \geq 1$

(σ2) $f(x) = x + \text{συν}x - 1$

$f'(x) = 1 - \text{ημ}x \geq 0$

$f \nearrow$

3.

(β) $f(x) = -x^2 - 2x + 3$

$f'(x) = -2x - 2$

$\longrightarrow f'(x) = 0 \iff -2x - 2 = 0$

$-2x = 2$

$\boxed{x = -1}$

x	$-\infty$		-1		$+\infty$
$f'(x)$		+	0	−	
$f(x)$		↗		↘	

Η f ↗ στο $(-\infty, -1]$

Η f ↘ στο $[-1, +\infty)$.

$A(-1, 4)$ O.M.

(δ) $f(x) = -x^3 + x^2 + x + 3$

$f'(x) = -3x^2 + 2x + 1$

x	$-\infty$		$-\frac{1}{3}$		1		$+\infty$
f'		−	0	+	0	−	
f		↘		↗		↘	

$\rightarrow f'(x) = 0 \iff -3x^2 + 2x + 1 = 0$

$\Delta = 4 + 12 = 16$

$x = \frac{-2 \pm 4}{-6}$ → ① , $\boxed{-\frac{1}{3}}$

4. (σε) $f(x) = x^4 + 2x^2 - 8x + 6$

$f'(x) = 4x^3 + 4x - 8 = 4\left(x^3 + x - 2\right) = 4(x-1)(x^2+x+2)$ ① $\Delta < 0$

①

1	0	1	-2
↓	1	1	2
1	1	2	0

x		1	
$4(x-1)$	−	0	+
x^2+x+2	+		+
f'	−		+
f	↘		↗

5. (γ) $f(x)=\frac{1}{5}x^5+\frac{1}{2}x^2-1$

$f'(x)=\frac{1}{5}5x^4+\frac{1}{2}2x=x^4+x=\underset{\textcircled{0}}{x}\underset{\textcircled{-1}}{(x^3+1)}$

$f'(x)=0 \Leftrightarrow x(x^3+1)=0$

$\boxed{x=0}$

$x^3+1=0$

$x^3=-1$

$\boxed{x=-1}$,

x		-1		0	
x	−		−	0	+
x^3+1	−	0	+		+
f'	+		−		+
f	↗		↘		↗

(ε) $f(x)=\frac{x^5}{5}-\frac{x^3}{3}+2x$

$f'(x)=\frac{1}{5}5x^4-\frac{1}{3}3x^2+2=x^4-x^2+2$

$f'(x)=0 \Rightarrow x^4-x^2+2=0 \Rightarrow t^2-t+2=0$

$t=1$ $\quad t=-2$

Θέτω $x^2=t$

$x^2=1$ $\quad x^2=-2$

$\boxed{x=1}$ $\boxed{x=-1}$ Αδύνατη

x		-1		1	
f'	+	0	+	0	+
f	↗		↗		↗

4. ⑧ $f(x) = \frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x - 1$

$f'(x) = \frac{1}{4} 4x^3 - 2 \cdot 3x^2 + \frac{11}{2} 2x - 6$

$f'(x) = x^3 - 6x^2 + 11x - 6 . = \underset{①}{(x-1)}\underset{② \; ③}{(x^2-5x+6)}$.

1	−6	11	−6	①
↓	1	−5	6	
1	−5	6	0	

x		1		2		3	
x−1	−	0	+		+		+
x^2-5x+6	+		+	0	−	0	+
f'	−		+		−		+
f	↘		↗		↘		↗

3. $\boxed{\sqrt{2}}$ $f(x)= x^3-6x+1$

$f'(x)= 3x^2-6 = 3(x^2-2)$

$\rightarrow f'(x)=0 \quad \Leftrightarrow 3(x^2-2)=0$

$x^2=2$

$\boxed{x=\pm\sqrt{2}}$

x		$-\sqrt{2}$		$\sqrt{2}$	
f'	+	0	−	0	+
f	↗		↘		↗

4. Ⓑ $f(x)= x^4+32x$

$f'(x)= 4x^3+32 = 4(x^3+8)$.

$\rightarrow f'(x)=0 \Rightarrow 4(x^3+8)=0 \Rightarrow x^3+8=0$

$x^3=-8$

$\boxed{x=-2}$

x		-2	
f'	−	0	+
f	↘		↗

7. Ⓑ $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{(x)' \cdot (x^2+1) - x \cdot (x^2+1)'}{(x^2+1)^2} = \frac{1 \cdot (x^2+1) - x \cdot 2x + 0}{(x^2+1)^2}$$

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\rightarrow f'(x) = 0 \iff \frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1-x^2 = 0.$$

$\iff x^2 = 1 \iff x = 1$ κ' $x = -1$

x		-1		1	
f'	$-$	0	$+$	0	$-$
f	↘		↗		↘

Ⓢ $f(x) = x + \frac{1}{x}$

$$f'(x) = (x)' + \frac{(1)' \cdot x - 1 \cdot (x)'}{x^2} = 1 + \frac{0-1}{x^2} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

$$f'(x) = 0 \iff \frac{x^2-1}{\underset{\oplus}{x^2}} = 0 \iff x = 1 \text{ κ } x = -1$$

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		↗		↘		↗	

$D_f = \mathbb{R}^*$

(02) $f(x) = x + \frac{4}{x^2}$

$f'(x) = (x)' + \frac{(4)' \cdot x^2 - 4 \cdot (x^2)'}{x^4} = 1 + \frac{0 - 8x}{x^4} = 1 - \frac{8x}{x^4}$

$= \frac{x^4 - 8x}{x^4} = \frac{x^3 - 8}{x^3}$

$f'(x) = 0 \iff \frac{x^4 - 8x}{\underset{\oplus}{x^4}} = 0 \iff x^4 - 8x = 0 \iff x^4 = 8x \iff$

$x^3 = 8 \iff \boxed{x = 2}$

$x^4 - 8x = 0$

$x(x^3 - 8) = 0$

$\boxed{x = 0}$ $x = 2.$

x	$-\infty$		0		2		$+\infty$
x^3-8		−	‖	−	0	+	
x^3		−	0	+		+	
$f'(x)$		+	‖	−	0	+	
f		↗	‖	↘		↗	

8. (Β) $f(x) = \ln|x| + \frac{1}{2x^2}$ $\qquad D_f = R^*$

$f'(x) = \frac{1}{x} + \frac{1}{2}2x$

$f'(x) = \frac{1}{x} + \frac{2x}{2}$

$f'(x) = \frac{1}{x} + x = \frac{1+x^2}{x}$

$\boxed{f'(x) = \frac{\overset{+}{1+x^2}}{x}}$

x		0	
f'	−	▨	+
f	↘	▨	↗

(δ) $f(x) = \frac{x^3-2x}{x^2-1}$, πρεπει $x^2-1 \neq 0$

$x^2 \neq 1$

$x \neq \pm 1$

$f'(x) = \frac{(x^3-2x)'(x^2-1) - (x^3-2x)(x^2-1)'}{(x^2-1)^2}$

$f'(x) = \frac{(3x^2-2)(x^2-1) - (x^3-2x) \cdot 2x}{(x^2-1)^2}$

$f'(x) = \frac{3x^4 - 3x^2 - 2x^2 + 2 - \cancel{3}2x^4 + 4x^2}{x^4-1}$

$f'(x) = \frac{x^4 - x^2 + 2}{x^4-1} \longrightarrow$ θετω $x^2 = t$

$t^2 - t + 2 = 0$

$\boxed{t=2}$ $\quad \boxed{t=-1}$

$\boxed{x=\pm\sqrt{2}}$ $\quad \boxed{x^2=2}$ $\quad x^2=-1$ Αδυν

x		$-\sqrt{2}$		-1		1		$\sqrt{2}$	
x^4-x^2+2	+	0	+	‖	+	‖	+	0	+
x^4-1	+		+	0	−	0	+		+
f'	+		+	‖	−	‖	+		+
f	↗		↗	‖	↘	‖	↗		↗

6. Ⓑ $f(x) = -\frac{4x^3}{3} + 2x^2 - x - 4$

$f'(x) = -\frac{4}{3}3x^2 + 4x - 1$

$f'(x) = -4x^2 + 4x - 1 = -(4x^2 - 4x + 1)$

$f'(x) = -(2x-1)^2 \leq 0$

$f \downarrow$

B' τρόπος

$f'(x) = 0$

$-4x^2 + 4x - 1 = 0$

$\Delta = 0$ $\boxed{x = \frac{1}{2}}$

x		$1/2$	
f'	$-$	0	$-$
f	↘		↘

(δ) $f(x) = x^4 - \frac{4}{3}x^3 - 2x^2 + 4x - 1$

x		-1		1	
$x-1$	$-$		$-$	0	$+$
x^2-1	$+$	0	$-$	0	$+$
f'	$-$		$+$		$-$
f	↘		↗		↘

$f'(x) = 4x^3 - \frac{4}{3}3x^2 - 2 \cdot 2x + 4$

$f'(x) = 4x^3 - 4x^2 - 4x + 4$

$f'(x) = 4(x^3 - x^2 - x + 1) = 4[x^2(x-1) - (x-1)]$

$f'(x) = 4(x-1)(x^2-1)$

① (−1) ①

9. Ⓑ $f(x)=\sqrt{4-x^2}$

AF: $4-x^2\geq 0$

$4\geq x^2$

$-2\leq x\leq 2 \Rightarrow AF=[-2,2]$

$$f'(x)=\frac{1}{2\sqrt{4-x^2}}\cdot(4-x^2)'=\frac{1}{2\sqrt{(2-x)(2+x)}}\cdot(-2x)=\frac{-2x}{2\sqrt{(2-x)(2+x)}}=\frac{-x}{\sqrt{(2-x)(2+x)}}$$

$$=\frac{-x}{\sqrt{4-x^2}}$$ [redacted] [redacted]

x	-2		0		2
f'		+	0	−	
f		↗		↘	

Ⓓ $f(x)=x-4\sqrt{x}$

AF: $[0,+\infty)$

$$f'(x)=x'-(4\sqrt{x})'=1-4.\frac{1}{2\sqrt{x}}=1-\frac{2}{\sqrt{x}}=\frac{\sqrt{x}-2}{\sqrt{x}}\ \oplus$$

$x-2\sqrt{x}=0$

$2\sqrt{x}=x$

$4x=x^2$

$\boxed{x=4}$

x	0		4		+∞
f'	0	−	0	+	
f		↘		↗	

(5c) $f(x) = x - 2\sqrt{x-2}$ AF: $x-2 \geq 0$

$x \geq 2$

$[2, +\infty)$

$$F'(x) = \cancel{(x-2)}'^{\,1}\sqrt{x-2} + (x-2)(\sqrt{x-2})'$$

$$F' = \sqrt{x-2} + (x-2)\cdot\frac{1}{2\sqrt{x-2}}\cdot\cancel{(x-2)}'^{\,1}$$

$$F' = \sqrt{x-2} + \frac{x-2}{2\sqrt{x-2}} = \cancel{2\sqrt{x-2}+}\ \frac{2(\sqrt{x-2})^2 + x-2}{2\sqrt{x-2}} = \frac{2(x-2)+x-2}{2\sqrt{x-2}}$$

$$= \frac{2x-4+x-2}{2\sqrt{x-2}} = \frac{3x-6}{2\sqrt{x-2}} = \frac{3\overset{2}{(x-2)}}{\underset{\oplus}{2\sqrt{x-2}}}$$

$\cancel{\frac{2\sqrt{x-2}}{8\sqrt{x-2}}}$

x	2	+∞
F	0	+
F'		↗

14. (δ) $f(x) = \dfrac{1}{1-\text{συν}x}$ $x \in (0, 2\pi)$.

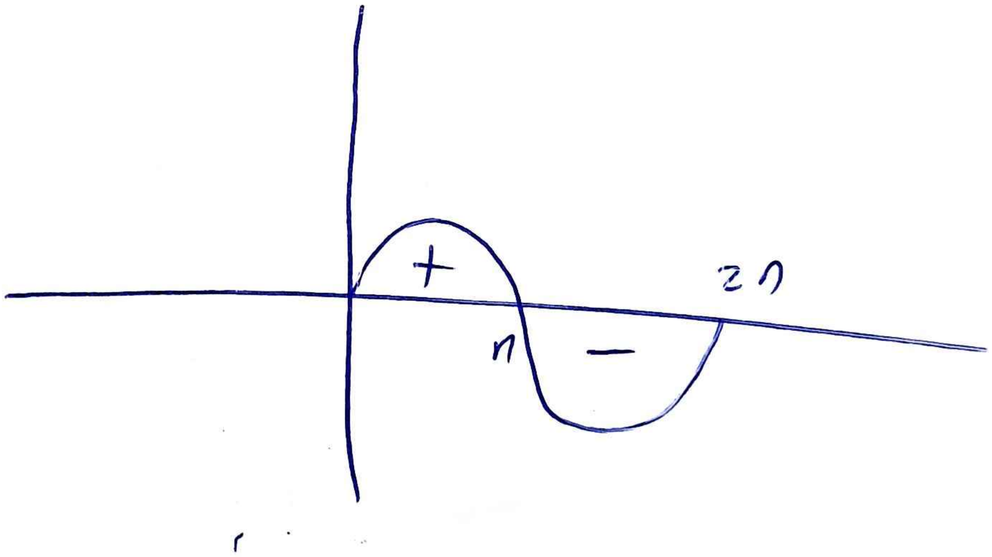
↳ [illegible]

$$f'(x) = \frac{\cancel{1'(1-\text{συν}x)}^{\to 0} - (1-\text{συν}x')\,1}{(1-\text{συν}x)^2} = \frac{\cancel{-}(-\text{ημ}x)}{\cancel{-(-\text{ημ}x)}\;(1-\text{συν}x)^2} = 1$$

$$f'(x) = \boxed{-}\,\frac{\text{ημ}x}{(1-\text{συν}x)^2}\,,$$

↓

x	0	π	2π
f'	−	+	
f	↘	↗	



10. Ⓑ $f(x) = e^x - e \cdot x$

$F'(x) = e^x - e$

$F'(x) = 0 \quad e^x - e = 0 \quad \Rightarrow e^x = e$

$\ln e^x = \ln e$

$\boxed{x = 1}$

X		1	
F'	—	0	+
F	↘		↗

✓

⑧ $f(x) = (x^2+x+1)\,e^x$

$F'(x) = (x^2+x+1)' \cdot e^x + (x^2+x+1) \cdot (e^x)'$

$F'(x) = (2x+1) \cdot e^x + (x^2+x+1)\,e^x$

$f'(x) = e^x\,(2x+1+x^2+x+1)$

$\boxed{f'(x) = e^x(x^2+3x+2)}$

$F'(x) = 0$

X		-2		-1	
F'	+	0	—	0	+
F	↗		↘		↗

(52) $f(x) = \dfrac{e^x}{x^2+1}$ $\qquad DF = \mathbb{R}$

$$F'(x) = \frac{(e^x)' \cdot (x^2+1) - e^x \cdot (x^2+1)'}{(x^2+1)^2}$$

$$F'(x) = \frac{e^x \cdot (x^2+1) - e^x \cdot (2x+1)}{(x^2+1)^2}$$

$$f'(x) = \frac{e^x(x^2+1-2x-1)}{(x^2+1)^2}$$

$$f'(x) = \frac{\overset{\oplus}{e^x}(x^2-2x)}{\underset{\oplus}{(x^2+1)^2}}$$

$\rightarrow x^2 - 2x = 0 \qquad \Rightarrow x(x-2) = 0$

$\boxed{x=0} \quad \boxed{x=2}$

x		0		2	
f'	+	0	−	0	+
f	↗		↘		↗

13. Ⓑ $f(x) = 2xe^{-x} + (x-1)^2$ $\quad DF = \mathbb{R}$

$F'(x) = (2x)' \cdot e^{-x} + 2x \cdot (e^{-x})' + 2(x-1)(x-1)'$

$F'(x) = 2e^{-x} - 2xe^{-x} + 2(x-1)$

$F'(x) =$

$f'(x) = 2e^{-x}(1-x) - 2(1-x)$

$$\boxed{f'(x) = (1-x)(2e^{-x}-2).}$$

$F'(x) = 0$

$1 - x = 0 \ \vee \ 2e^{-x} - 2 = 0$

$\boxed{x = 1}$

$\frac{2e^{-x}}{2} = \frac{2}{2}$

$e^{-x} = 1$

$\frac{1}{e^x} = 1$

✓

$e^x = 1$

$\ln e^x = \ln 1$

$\boxed{x = 0}$

~~x~~		~~0~~		~~1~~	
~~F'(~~	~~+~~	~~0~~	~~−~~	~~0~~	~~+~~
~~F~~	~~↗~~		~~↘~~		~~↗~~

x		0		1	
1−x	+		+	0	−
$2e^{-x}-2$	+	0	−		−
f'	+		−		+
f	↗		↘		↗

13. ⑧ . $f(x)=(x^2-x)\ln x+\frac{x^2}{2}$

AF: $x>0$: $(0,+\infty)$

$$f'(x)=\Big((x^2-x)'\ln x+(\ln x)'(x^2-x)\Big)+\frac{1}{2}(x^2)'$$

$$f'=(2x-1)\cdot\ln x+\frac{1}{\cancel{x}}\cdot\cancel{x}(x-1)+\frac{1}{\cancel{2}}\cdot\cancel{2}x$$

$$f'(2x-1)\ln x+x-1+x$$

$$f'=(2x-1)\cdot\ln x+2x-1 \quad =(2x-1)(\ln x+1)$$

~~$f(x)=(2x-1)(\ln x+1)$~~

~~$x=0$~~

~~x | 0 | +∞~~

~~f' | 0 | +~~

~~f | ↗~~

$2x-1=0$

$2x=1$

$$\boxed{x=\frac{1}{2}}$$

$\ln x+1=0$

$\ln x=-1$

$\ln x=-\ln e$

$$\boxed{x=-e}$$

↳ απορ

x	0		$\frac{1}{2}$	+∞
$2x-1$		−	0	+
$\ln x+1$		+		+
f'		−	0	+
f		↘		↗

9. (σΣ) $f(x) = x - 2\sqrt{x-2}$ Af: $[2,+\infty)$

$$F'(x) = (x)' - 2\left(\sqrt{x-2}\right)'$$

$$F' = 1 - \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x-2}}\,\cancel{(x-2)'} \to 1$$

$$F' = 1 - \frac{1}{\sqrt{x-2}} = \frac{\sqrt{x-2} - 1}{\underset{\substack{\uparrow\\ x=2}}{\sqrt{x-2}}} \leftarrow x=3$$

$\sqrt{x-2} - 1 = 0$

$\sqrt{x-2} = 1$

$x - 2 = 1$

$\boxed{x=3}$

~~X~~	~~2~~	~~3~~	~~+oo~~
~~F~~	~~0~~	~~0~~	
~~F'~~			

X	2		3		+oo
$\sqrt{x-2}-1$		−	0	+	
$\sqrt{x-2}$	0	+		+	
F'		−		+	
F		↘		↗	

11. Ⓑ $f(x) = xe^{\frac{1}{x}}$ $D_f = \mathbb{R}^*$.

$f'(x) = (x)' \cdot (e^{\frac{1}{x}}) + (x) \cdot (e^{\frac{1}{x}})'$

$f'(x) = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} (\frac{1}{x})'$

$f'(x) = e^{\frac{1}{x}} + xe^{\frac{1}{x}} \cdot \frac{-1}{x^2}$

$f'(x) = e^{\frac{1}{x}} - \frac{xe^{\frac{1}{x}}}{x^2}$

$f'(x) = e^{\frac{1}{x}} - \frac{e^{\frac{1}{x}}}{x}$

$f'(x) = \frac{x \cdot e^{\frac{1}{x}} - e^{\frac{1}{x}}}{x}$

$f'(x) = \frac{e^{1/x}(x-1)}{x} \to ①$

$x \downarrow \boxed{0}$

	$-\infty$		0		1		$+\infty$
$x-1$		$-$		$-$	0	$+$	
x		$-$	0	$+$		$+$	
f'		$+$		$-$		$+$	
f		↗		↘		↗	

(δ) $f(x) = e^x - e^{-x} - 2x$

$f'(x) = e^x - e^{-x}(-x)' - 2$

$f'(x) = e^x + e^{-x} - 2 = e^x + \frac{1}{e^x} - 2 = \frac{e^{2x} + 1 - 2e^x}{e^x}$

$f'(x) = \frac{e^{2x} - 2e^x + 1}{e^x} = \frac{(e^x - 1)^2}{e^x} \geqslant 0$ $f\nearrow$

✓

12. ⓑ $f(x) = x^2 \ln x$

$$f'(x) = (x^2)' \cdot \ln x + x^2 \cdot (\ln x)' = 2x \cdot \ln x + \frac{x^{\cancel{2}}}{\cancel{x}} = 2x\ln x + x$$

$$f'(x) = x\,(2\ln x + 1).$$

$\boxed{x=0}$ ή $2\ln x + 1 = 0 \iff \ln x = -\frac{1}{2} \iff \boxed{x = e^{-\frac{1}{2}}}$

$$x = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}}$$

x		0		$\frac{1}{\sqrt{e}}$	
x	$-$	0	$+$		$+$
$2\ln x + 1$	$-$		$-$	0	$+$
$f'(x)$	$+$		$-$		$+$
$f(x)$	↗		↘		↗

ⓓ. $f(x) = \dfrac{\ln x}{\sqrt{x}}$ $\quad D_f = (0, +\infty)$.

$$f'(x) = \frac{(\ln x)' \cdot \sqrt{x} - \ln x \cdot (\sqrt{x})'}{(\sqrt{x})^2} = \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{2\sqrt{x}^2 - x\ln x}{2x\sqrt{x}\,(\sqrt{x})^2} = \frac{2x - x\ln x}{2x^2\sqrt{x}} = \frac{2 - \ln x}{2x\sqrt{x}}.$$

$\ln x = 2 \iff x = e^2$

x	0		e^2	
$2 - \ln x$	▨	$+$	0	$-$
$f'(x)$	▨	$+$		$-$
$f(x)$	▨	↗		↘

13. (σε) $f(x) = \dfrac{x}{\ln^2 x}$

$\boxed{x>0}$

$x \neq 1$

$$f'(x) = \frac{(x')\cdot \ln^2 x - \cancel{x}\cdot 2\ln x \cdot \frac{1}{\cancel{x}}}{\ln^4 x}$$

$$= \frac{\ln^2 x - 2\ln x}{\ln^4 x} = \frac{\ln x - 2}{\ln^3 x}$$

x	0	1		e^2	
$\ln x - 2$	$-$	‖	$-$	0	$+$
$\ln^3 x$	$-$	‖	$+$		$+$
f'	$+$	‖	$-$		$+$
f	↗	‖	↘		↗

14. Ⓑ $f(x) = \frac{x^2}{2} - x\,\text{συν}x + \text{ημ}x.$

$$f'(x) = \frac{1}{2}(x^2)' - \Big((x)'\,\text{συν}x + x(\text{συν}x)'\Big) + \text{συν}x$$

$$f'(x) = \frac{2x}{2} - \text{συν}x + x\cdot(-\text{ημ}x) + \text{συν}x$$

$$f'(x) = x + x(-\text{ημ}x)$$

$$f'(x) = x\,(1 \overset{\oplus}{+} \text{ημ}x)$$

$\downarrow$

$\boxed{0}$

	-∞		0		$\frac{\pi}{2}$		+∞
x		−	0	+		+	
$1-\text{ημ}x$		+		+	0	+	
f'		−		+		+	
f		↗		↗		↗	

28. ⓐ $f(x) = \begin{cases} (x-2)e^{x-1}+2, & x<1 \\ -x^2+2x, & x \geq 1, \end{cases}$

$\lim_{x \to 1-} f(x) = (x-2)e^{x-1}+2 = 1$

$\lim_{x \to 1+} f(x) = -x^2+2x = 1$

το οριο υπάρχει
άρα συνεχής στο 1
και συνεχής στο πεδίο ορισμού της
ως π.σ.σ.

$\underline{x<1}$

$f_1(x) = (x-2)e^{x-1}+2$

$f'_1(x) = (x-2)' \cdot e^{x-1} + (e^{x-1})'(x-2)$

$f'_1(x) = e^{x-1} + e^{x-1}(x-1)'(x-2)$

$f'_1(x) = e^{x-1} + e^{x-1}(x-2) = e^{x-1}(1+x-2) = e^{x-1}(x-1)$

$\downarrow$

$\boxed{x=1}$

$x>1$

$f_2(x) = -x^2+2x$

$f'(x) = -2x+2 = 2(1-x)$

$\downarrow$

$\boxed{x=1}$

x		1	
f_1'	−	0	+
f_2'	+	0	−
F'	−	0	−
F	↘		↘

✓

28. Ⓑ $f(x) = \begin{cases} ημx , & -π < x \leq 0 \\ x^2 \ln x , & x > 0 . \end{cases}$

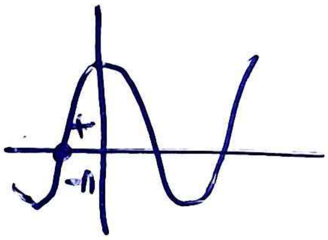
$\lim_{x \to 0^-} f(x) = 0$ } το οριο υπαρχει και $f(0)=0$ αρα $\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0^+} f(x) = 0$ οποτε συνεχης

$a < x \leq 0$

$f(x) = ημx$

$f'(x) = συνx \geq 0$



$x > 0$

$f(x) = x^2 \ln x$

$f'(x) = (x^2)' \ln x + x^2 (\ln x)'$

$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$

$f'(x) = 2x \cdot \ln x + \frac{x^2}{x}$

$f'(x) = 2x \ln x + x$

$f'(x) = x(2\ln x + 1)$

$\downarrow$

0

$2\ln x + 1 = 0$

$2\ln x = -1$

$\ln x = -\frac{1}{2}$

$\boxed{x = \frac{\sqrt{e}}{e}}$

	$-π$	0	$\frac{\sqrt{e}}{e}$ $+\infty$
~~f_1'~~	~~+~~	~~+~~	~~+~~
~~f_2'~~	~~+~~	~~−~~	~~+~~
~~f'~~	~~+~~	~~−~~	~~+~~
~~f~~	~~↗~~	~~↘~~	~~↗~~

x	$-\pi$		0		$\frac{\sqrt{e}}{e}$	
f_1'		$+$	‖	////		//
f_2'		//////	‖	$-$	0	$+$
f'		$+$	‖	$-$		$+$
f		$\nearrow$	‖	$\searrow$		$\nearrow$

28. (γ) $f(x) = \begin{cases} \ln(e^x - x), & x \le 0 \\ \ln x - x, & x > 0 \end{cases}$,

$\cdot \lim\limits_{x \to 0^-} F(x) = 0 = \lim\limits_{x \to 0^-} \ln(e^x - x) = 0$

$\cdot \lim\limits_{x \to 0^+} F(x) = 0 = \lim\limits_{x \to 0^+} \ln x - x = -\infty$

Το $\lim\limits_{x \to 0} F(x)$ δεν υπάρχει

Άρα δεν είναι συνεχης στο 0 και γενικα!

$x \le 0$

$F_1(x) = \ln(e^x - x)$

$F_1'(x) = \frac{1}{e^x - x}(e^x - x)'$

$F_1'(x) = \frac{e^x - 1}{e^x - x}$.

$F_1'(x) = 0 \Rightarrow e^x - 1 = 0$

$e^x = 1$

$\boxed{x = 0}$

$x > 0$

$F_2(x) = \ln x - x$

$F_2'(x) = \frac{1}{x} - 1$

$\boxed{F_2'(x) = -\frac{1-x}{x}}$ ①

X		0		1	
F_1'	−		~~+~~		~~+~~
F_2'	~~+~~		+	0	−
F'	−		+	0	−
F	↘		↗		↘

28. (δ) $f(x) = \begin{cases} x^2 e^{1-x} + 2 & x<1 \\ x-2\sqrt{x-1}, & x \geq 1. \end{cases}$

$$\left.\begin{aligned} &\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2\cdot e^{1-x} + 2 = 3 \\ &\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x - 2\sqrt{1-1} = 1 \end{aligned}\right\}$$ Δεν είναι συνεχής.

$x<1$

$f_1(x) = x^2\cdot e^{1-x} + 2$

$f_1'(x) = 2x\cdot e^{1-x} + x^2\cdot e^{1-x}\cdot(-1) + 0 = 2x\cdot e^{1-x} - x^2\cdot e^{1-x}$

$f_1'(x) = e^{1-x}\cdot(2x-x^2) = e^{1-x}x(2-x)$

⓪ ②

$x \geq 1$

$f_2(x) = x - 2\sqrt{x-1}$

$f_2'(x) = 1 - 0 + 2\cdot(\sqrt{x-1})' = 1 - \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x-1}}\cdot(x-1)'$

$= 1 - \frac{1}{\sqrt{x-1}} = \frac{\sqrt{x-1}-1}{\sqrt{x-1}}$

$x=1$

$x=2$

.85

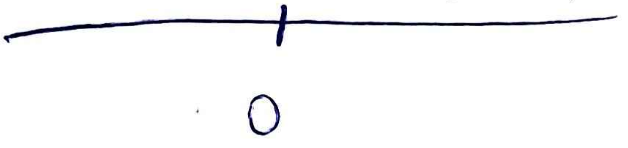
x		0		1		2	
f_1'	−	0	+	‖	~~+~~	~~0~~	~~−~~
f_2'	▨	▨	▨	‖	−	0	+
f'	−		+	‖	−		+
f	↘		↗	‖	↘		↗

16. Ⓑ $f(x) = \begin{cases} x^3 - 3x, & x < 0 \\ x^2 - 4x, & x \geq 0 \end{cases}$

- $\lim_{x \to 0^-} f(x) = 0$
- $\lim_{x \to 0^+} f(x) = 0$

$\left.\right\} \lim_{x \to 0} f(x) = 0 = f(0)$

✓ Συνεχής.



$x < 0$

$f_1(x) = x^3 - 3x$

$f_1'(x) = 3x^2 - 3$

$f_1'(x) = 3(x^2 - 1)$.

(−1) ①

$x \geq 0$

$f_2(x) = x^2 - 4x$

$f_2'(x) = 2x - 4$

$f_2'(x) = 2(x - 2)$

②

x		-1		0		1		2	
f_1'	+	0	−		−	0	+		+
f_2'	+		+		−		−	0	+
f'	+		−		−		−		+
f	↗		↘		↘		↘		↗

ΑΣΚΗΣΕΙΣ ΓΙΑ ΛΥΣΗ

Α. Εύρεση μονοτονίας αλγεβρικά

2. Να μελετήσετε τις παρακάτω συναρτήσεις ως προς τη μονοτονία.

α. $f(x)=e^x+x^3-1$

~~**β.**~~ $f(x)=\frac{1}{x}-\ln x$

γ. $f(x)=-x^3+x^2-x+1$

~~**δ.**~~ $f(x)=2x+συνx-1$

ε. $f(x)=1+(x-2)^3$

~~**στ.**~~ $f(x)=x+συνx-1$

3. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=x^2-3x+1$

~~**β.**~~ $f(x)=-x^2-2x+3$

γ. $f(x)=\frac{1}{3}x^3-2x^2+3x+1$

~~**δ.**~~ $f(x)=-x^3+x^2+x+3$

ε. $f(x)=x^3-3x+1$

~~**στ.**~~ $f(x)=x^3-6x+1$

4. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\frac{x^4}{4}-x+1$

~~**β.**~~ $f(x)=x^4+32x$

γ. $f(x)=\frac{1}{4}x^4-2x$

~~**δ.**~~ $f(x)=\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x-1$

ε. $f(x)=-\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2-2x$

~~**στ.**~~ $f(x)=x^4+2x^2-8x+6$

5. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=x^4-2x^2-1$

β. $f(x)=5x-x^5$

~~**γ.**~~ $f(x)=\frac{1}{5}x^5+\frac{1}{2}x^2-1$

δ. $f(x)=\frac{1}{5}x^5-x^4-\frac{1}{2}x^2+4x$

~~**ε.**~~ $f(x)=\frac{x^5}{5}-\frac{x^3}{3}+2x$

6. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\frac{1}{3}x^3-x^2+x-1$

~~**β.**~~ $f(x)=-\frac{4x^3}{3}+2x^2-x-4$

γ. $f(x)=-3x^4+4x^3+2$

~~**δ.**~~ $f(x)=x^4-\frac{4}{3}x^3-2x^2+4x-1$

ε. $f(x)=(x-1)^3(x-3)^3$

~~**στ.**~~ $f(x)=(x+1)^5(x-2)^4$

7. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\frac{x}{x+1}$

~~**β.**~~ $f(x)=\frac{x}{x^2+1}$

γ. $f(x)=\frac{x}{x^2-1}$

~~**δ.**~~ $f(x)=x+\frac{1}{x}$

ε. $f(x)=x^2+\frac{2}{x}$

~~**στ.**~~ $f(x)=x+\frac{4}{x^2}$

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8. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\frac{x}{(x-1)^2}$ β. $f(x)=\ln|x|+\frac{1}{2x^2}$

γ. $f(x)=\frac{2-x^2}{x^4}$ δ. $f(x)=\frac{x^3-2x}{x^2-1}$

9. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\sqrt{x^2+4}$ β. $f(x)=\sqrt{4-x^2}$ γ. $f(x)=\sqrt{x^2-9}$

δ. $f(x)=x-4\sqrt{x}$ ε. $f(x)=\sqrt{x}+\sqrt{2-x}$ στ. $f(x)=x-2\sqrt{x-2}$

10. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=e^x-x+3$ β. $f(x)=e^x-ex$ γ. $f(x)=(x-2)e^x$

δ. $f(x)=(x^2+x+1)e^x$ ε. $f(x)=\frac{x^2}{e^x}$ στ. $f(x)=\frac{e^x}{x^2+1}$

11. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=e^{-2x}+x-1$ β. $f(x)=x\cdot e^{\frac{1}{x}}$

γ. $f(x)=e^{2x}-3e^x+x-1$ δ. $f(x)=e^x-e^{-x}-2x$

12. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=x\ln x-2x$ β. $f(x)=x^2\ln x$

γ. $f(x)=\frac{\ln x}{x}$ δ. $f(x)=\frac{\ln x}{\sqrt{x}}$

ε. $f(x)=x-e\ln x$ στ. $f(x)=x+\ln(x^2+1)$

ζ. $f(x)=x\ln^2 x-3x+2$ η. $f(x)=x^{\ln x}\ ,\ x>0$

13. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=(x-2)e^x-\frac{1}{2}x^2+x-1$ β. $f(x)=2xe^{-x}+(x-1)^2$

γ. $f(x)=(x^2-4x)\ln x-\frac{x^2}{2}+4x$ δ. $f(x)=(x^2-x)\ln x+\frac{x^2}{2}$

ε. $f(x)=\frac{e^x}{x^2}$ στ. $f(x)=\frac{x}{\ln^2 x}$

14. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=e^{\eta\mu x}\ ,\ x\in[0\,,\,2\pi]$ β. $f(x)=\frac{x^2}{2}-x\sigma\upsilon\nu x+\eta\mu x$

γ. $f(x)=\eta\mu^2 x+4\sigma\upsilon\nu x\ ,\ x\in[-\pi\,,\,\pi]$ δ. $f(x)=\frac{1}{1-\sigma\upsilon\nu x}\ ,\ x\in(0,2\pi)$

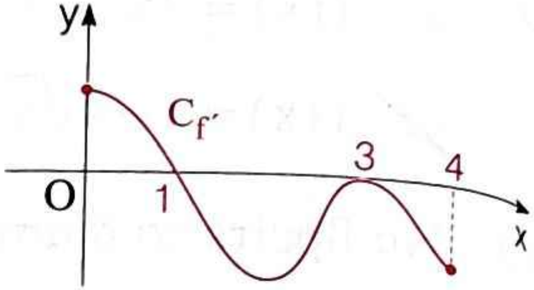
15. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\dfrac{ημx}{1+συνx}\ ,\ \ x\in(-\pi\ ,\ \pi)$ **β.** $f(x)=ημx-εφx\ ,\ \ x\in\left(-\dfrac{\pi}{2}\ ,\ \dfrac{\pi}{2}\right)$

16. Να βρείτε τα διαστήματα μονοτονίας των συναρτήσεων:

α. $f(x)=\begin{cases} x^2-2x+3\ , & x<1 \\ \dfrac{1}{x}+1 & ,\ x\geq 1 \end{cases}$ **β.** $f(x)=\begin{cases} x^3-3x\ , & x<0 \\ x^2-4x\ , & x\geq 0 \end{cases}$

17. Στο διπλανό σχήμα φαίνεται η γραφική παράσταση της παραγώγου μιας συνεχούς συνάρτησης $f:[0\ ,4]\rightarrow\mathbb{R}$. Να βρείτε τα διαστήματα μονοτονίας της f.



B1. Εύρεση μονοτονίας συνάρτησης με τη βοήθεια της f''

18. Να βρείτε, τις ρίζες και το

4. ⓔ $f(x) = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 - 2x$

$$f'(x) = -\frac{1}{4}4x^3 + \frac{2}{3}3x^2 + \frac{1}{2}2x - 2$$

$$f'(x) = -x^3 + 2x^2 + x - 2$$

$$f'(x) = -x^2(x-2) + x - 2$$

$$\boxed{f'(x) = (\underset{\textcircled{2}}{x-2})(\underset{\textcircled{1}\ \textcircled{-1}}{1-x^2}).}$$

x		-1		1		2	
x-2	−		−		−	0	+
$1-x^2$	−	0	+	0	−		−
f'	+		−		+		−
f	↗		↘		↗		↘

5. ⓐ . $f(x) = x^4 - 2x^2 - 1$

$$f'(x) = 4x^3 - 4x = 4x(x^2-1)$$

$$\boxed{f'(x) = 4x(x^2-1)}$$

x		-1		0		1	
4.x	−		−	0	+		+
x^2-1	+	0	−		−	0	+
f'	−		+		−		+
f	↘		↗		↘		↗

5. (β) $f(x) = 5x - x^5$

$$f'(x) = 5 - 5x^4 = 5(1 - x^4) = 5(1-x^2)(1+x^2)$$

$$\boxed{f'(x) = 5\underset{-1\ \ 1}{(1-x^2)}\underset{+}{(x^2+1)}}$$

x		-1		1	
f'	$-$	0	$+$	0	$-$
f	↘		↗		↘

(δ) $f(x) = \frac{1}{5}x^5 - x^4 - \frac{1}{2}x^2 + 4x$

$$f'(x) = \frac{1}{5}5x^4 - 4x^3 - \frac{1}{2}2x + 4$$

$$f'(x) = x^4 - 4x^3 - x + 4$$

$$f'(x) = x^3(x-4) - (x-4) = (x^3-1)(x-4).$$

$$\boxed{f'(x) = \underset{1}{(x-1)}\underset{+}{(x^2+x+1)}\underset{4}{(x-4)}}$$

x		1		4	
$x-1$	$-$	0	$+$		$+$
$x-4$	$-$		$-$	0	$+$
f'	$+$		$-$		$+$
f	↗		↘		↗

6. (٢) $f(x) = -3x^4 + 4x^3 + 2$

$$f'(x) = -12x^3 + 12x^2 = \underset{⓪}{-12x^2}\underset{①}{(x-1)}.$$

x		0		1	
$-12x^2$	$-$	0	$-$		$-$
$x-1$	$-$		$-$	0	$+$
f'	$+$		$+$		÷
f	↗		↗		↘

(٤) $f(x) = (x-1)^3(x-3)^3$

$$f'(x) = 3(x-1)^2(x-3)^3 + (x-1)^3\,3(x-3)^2$$

$$f'(x) = 3\,(x-1)^2(x-3)^2\,(x-3+x-1)$$

$$\boxed{f'(x) = 3\underset{①}{(x-1)^2}\underset{③}{(x-3)^2}\underset{②}{(2x-4)}}$$

x		2	
f'	$-$	0	$+$
f	↘		↗

7. ⓑ $f(x) = \dfrac{x}{x^2-1}$

$D_f = \mathbb{R} - \{1, -1\}.$

$$f'(x) = \frac{x^2-1 - x\cdot 2x}{(x^2-1)^2} = \frac{\overset{\ominus}{-1-x^2}}{\underset{\oplus}{(x^2-1)^2}} < 0$$

$f \downarrow$

8. ⓐ $f(x) = \dfrac{x}{(x-1)^2}$

$$f'(x) = \frac{(x-1)^2 - x\cdot 2(x-1)}{(x-1)^4} = \frac{x-1-2x}{(x-1)^3}$$

$$\boxed{f'(x) = \frac{\overset{(-1)}{-x-1}}{\underset{(1)}{(x-1)^3}}}$$

x		-1		1	
$-x-1$	$+$	0	$-$	‖	$-$
$(x-1)^3$	$-$		$-$	‖	$+$
f'	$-$		$+$	‖	$-$
f	↘		↗	‖	↘

8. ① $f(x)=\frac{2-x^2}{x^4}$ $D_f=\mathbb{R}^*$.

$$f'(x)=\frac{-2x\cdot x^4-(2-x^2)\,4x^3}{x^8}=\frac{-2x^2-4(2-x^2)}{x^5}$$

$$f'(x)=\frac{-2x^2-8+4x^2}{x^5}=\frac{2x^2-8}{x^5}=\frac{2(x^2-4)}{x^5}$$

② (−2)

$$\boxed{f'(x)=2\,\frac{x^2-4}{x^5}}$$

⓪

x		-2		0		2	
x^2-4	$+$	0	$-$	▨	$-$	0	$+$
x^5	$-$		$-$	0	$+$		$+$
f'	$-$		$+$	▨	$-$		$+$
f	↘		↗	▨	↘		↗.

9. ⓔ $f(x) = \sqrt{x} + \sqrt{2-x}$

πρπ $x \geq 0$ και $2 - x \geq 0$

$x \leq 2$.

$D_f = [0,2]$.

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{2-x}} = \frac{\sqrt{2-x} - \sqrt{x}}{2\sqrt{x}\sqrt{2-x}}$$

$\longrightarrow \sqrt{2-x} - \sqrt{x} = 0$

$\sqrt{2-x} = \sqrt{x}$

$2 - x = x$

$2 = 2x$ $\boxed{x = 1}$.

x	0		1		2
f'		+	0	−	
f		↗		↘	

10. (γ) $f(x) = (x-2)e^x$

$$f'(x) = e^x + (x-2)e^x = e^x(1+x-2) = e^x(x-1)$$

$$\boxed{f'(x) = e^x \underbrace{(x-1)}_{①}}$$

x		1	
f'	−	0	+
f	↘		↗

(ε) $f(x) = \dfrac{x^2}{e^x}$ $\quad D_f = \mathbb{R}$.

$$f'(x) = \frac{2xe^x - x^2e^x}{(e^x)^2} = \frac{2x - x^2}{e^x}$$

$$f'(x) = \frac{\overset{⓪}{x}\,\overset{②}{(2-x)}}{e^x}$$

x		0		2	
f'	−	0	+	0	−
f	↘		↗		↘

11. (01) $f(x) = e^{-2x} + x - 1$

$$f'(x) = -2e^{-2x} + 1$$

$\longrightarrow f'(x) = 0 \quad \Rightarrow -2e^{-2x} + 1 = 0$

$$1 = 2e^{-2x}$$

$$e^{-2x} = \frac{1}{2}$$

$$-2x = \ln \frac{1}{2}$$

$$-2x = \ln 1 - \ln 2$$

$$-2x = -\ln 2$$

$$\boxed{x = \frac{\ln 2}{2}}$$

x		$\frac{\ln 2}{2}$	
f'	$-$	0	$+$
f	↘		↗

$$f'(\ln 2) = -2e^{-2\ln 2} + 1 = -2\frac{1}{e^{2\ln 2}} + 1 = -2\frac{1}{e^{\ln 4}} + 1$$

$$= -\frac{1}{2} + 1 > 0$$

11. (γ). $f(x) = e^{2x} - 3e^{x} + x - 1$

$f'(x) = 2e^{2x} - 3e^{x} + 1$

$f'(x) = 0 \Rightarrow 2e^{2x} - 3e^{x} + 1 = 0$

$\boxed{e^{x} = t}$

x		$-\ln 2$		0	
f'	+	0	−	0	+
f	↗		↘		↗

$2t^2 - 3t + 1 = 0$

$\Delta = 9 - 8 = 1$

$t = \frac{3 \pm 1}{4}$

$\to t = 1 \Rightarrow e^{x} = 1$ $\boxed{x = 0}$

$\to t = \frac{1}{2} \Rightarrow e^{x} = \frac{1}{2}$ $\boxed{x = -\ln 2}$

12. (γ) $f(x) = \dfrac{\ln x}{x}$ $\quad D_f = (0,+\infty)$.

$$f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$\longrightarrow 1-\ln x = 0 \Rightarrow 1 = \ln x \Rightarrow \underline{\underline{x=e}}$$

x	0		e		+∞
f'		+	0	−	
f		↗		↘	

(ε). $f(x) = x - e\ln x$ $\quad D_f = (0,+\infty)$.

$$f'(x) = 1 - e\frac{1}{x} = \frac{x-e}{x}$$

x	0		e	
f'		−	0	+
f		↘		↗

12. (02) $f(x) = x + \ln(x^2+1))$

$$f'(x) = 1 + \frac{2x}{x^2+1} = \frac{x^2+1+2x}{x^2+1} = \frac{(x+1)^2}{x^2+1} \geq 0$$

$f \nearrow$

① $f(x) = x\ln^2 x - 3x + 2.$ $D_f = (0,+\infty).$

$$f'(x) = \ln^2 x + x \cdot 2\ln x \, \frac{1}{x} - 3$$

$$\boxed{f'(x) = \ln^2 x + 2\ln x - 3 \,.}$$

$\rightarrow \ln^2 x + 2\ln x - 3 = 0 .$

$\Delta = 4 + 12 = 16$

$$\ln x = \frac{-2 \pm 4}{2} \begin{cases} \rightarrow \ln x = 1 \Rightarrow \boxed{x = e} \\ \rightarrow \ln x = -3 \quad \boxed{x = e^{-3}} . \end{cases}$$

x		$\frac{1}{e^3}$		e	
f'	$+$	0	$-$	0	$+$
f	$\nearrow$		$\searrow$		$\nearrow$

ⓝ . $f(x) = x^{\ln x}$, $x > 0$

$$f(x) = e^{\ln x^{\ln x}} = e^{\ln x \ln x}$$

$$f'(x) = e^{\ln x \ln x}\left(\frac{1}{x}\ln x + \ln x \frac{1}{x}\right)$$

$$f'(x) = x^{\ln x}\, 2\frac{\ln x}{x}.$$

$\rightarrow \ln x = 0 \Rightarrow \underline{\underline{x = 1}}$

		1	
f'	−	0	+
f	↘		↗ .

13. ⓐ $f(x) = (x-2)\,e^{x} - \frac{1}{2}x^{2} + x - 1$.

$$f'(x) = e^{x} + (x-2)\,e^{x} - \frac{1}{2}\,2x + 1$$

$$f'(x) = e^{x}(1+x-2) - x + 1$$

$$f'(x) = e^{x}(x-1) - (x-1)$$

$$\boxed{f'(x) = \underset{\textcircled{1}}{(x-1)}\;\underset{\textcircled{0}}{(e^{x}-1)}\,.}$$

x		0		1	
$x-1$	−		−	0	+
$e^{x}-1$	−	0	+		+
f'	+		−		+
f	↗		↘		↗.

13. (γ) $f(x) = (x^2-4x)\ln x - \frac{x^2}{2} + 4x$

$D_f = (0, +\infty)$.

$$f'(x) = (2x-4)\ln x + (x^2-4x)\frac{1}{x} - \frac{1}{2}2x + 4$$

$$f'(x) = (2x-4)\ln x + \cancel{x} \cancel{-4} - \cancel{x} + \cancel{4}$$

$$\boxed{f'(x) = \underset{\textcircled{2}}{(2x-4)}\underset{\textcircled{1}}{\ln x}}$$

x	0		1		2		+∞
2x−4		−		−	0	+	
ln x		−	0	+		+	
f′		+		−		+	
f		↗		↘		↗	

13. (ε). $f(x) = \frac{e^x}{x^2}$ $\quad D_f = \mathbb{R}^*$

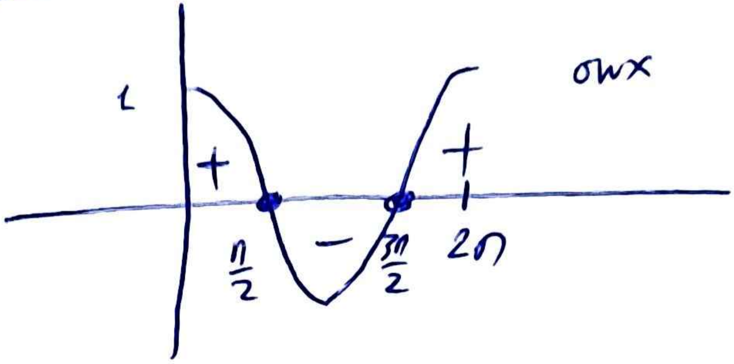
$$f'(x) = \frac{e^x x^2 - e^x 2x}{x^4} = \frac{xe^x - 2e^x}{x^3} = \frac{e^x(x-2)}{x^3}$$

$$\boxed{f'(x) = \frac{e^x(x-2)}{x^3}}$$

x		0		2	
$x-2$	−		−	0	+
x^3	−	0	+		+
f'	+		−		+
f	↗		↘		↗

14. ⓐ. $f(x) = e^{ημx}$ $\qquad x \in [0, 2π]$.

$$\boxed{f'(x) = e^{ημx}\, συνx.}$$



x	0		$\frac{π}{2}$		$\frac{3π}{2}$		2π
f'		+	0	−	0	+	
f		↗		↘		↗	

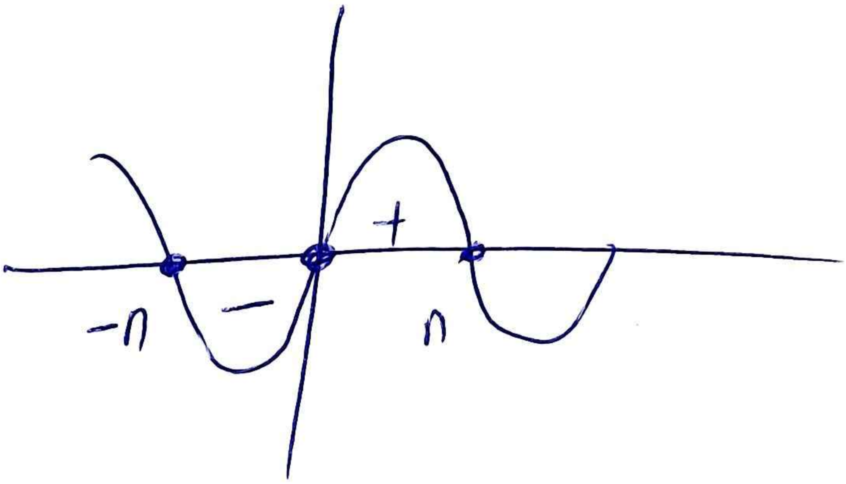
14. (γ) $f(x)=\eta\mu^2 x+4\,\sigma\upsilon\nu x \quad x\in[-\pi,\pi]$.

$$f'(x)=2\eta\mu x\,\sigma\upsilon\nu x-4\eta\mu x$$

$$f'(x)=2\eta\mu x(\underset{\ominus}{\sigma\upsilon\nu x-2})$$

$$\longrightarrow \quad -1\le \sigma\upsilon\nu x\le 1$$

$$\underline{\underline{-3\le \sigma\upsilon\nu x-2\le -1}}$$



x	−π		0		π
$2\eta\mu x$		−	0	+	
$\sigma\upsilon\nu x-2$		−		−	
f'		+		−	
f		↗		↘	

16. ⓐ $f(x) = \begin{cases} x^2-2x+3, & x<1 \\ \frac{1}{x}+1, & x\geq 1. \end{cases}$

$\left.\begin{array}{l} \cdot \lim\limits_{x\to 1^-} f(x) = 2 \\ \cdot \lim\limits_{x\to 1^+} f(x) = 2 \end{array}\right\} \lim\limits_{x\to 2} f(x) = 2 = f(2)$

Συνεχής στο 1.

$x<1$

$f_1(x) = x^2-2x+3$

$f_1'(x) = 2x-2$

$f_1'(x) = 2(\underset{①}{x-1})$

$x \geq 1$

$f_2(x) = \frac{1}{x}+1$

$f_2'(x) = -\frac{1}{x^2} < 0$

x		1	
f_1'	−		▨ +
f_2'	▨ +		−
f'	−		−
f	↘		↘

20. (γ) $f(x) = e^{x-1} + x\ln x - 2x$

$D_f = (0, +\infty)$

$f'(x) = e^{x-1} + \ln x + x\,\frac{1}{x} - 2$

$\boxed{f'(x) = e^{x-1} + \ln x - 1}$ $\qquad f'(1) = 0$

$f''(x) = e^{x-1} + \frac{1}{x} > 0$

x	0	1	$+\infty$
f''	+		+
f'	↗ −	0	+ ↗
f	↘		↗

(δ) $f(x) = e^{x} - \frac{x^3}{6} - \frac{x^2}{2} - x - 1$

$f'(x) = e^{x} - \frac{x^2}{2} - x - 1 \qquad \underline{\underline{f'(0) = 0}}$

$f''(x) = e^{x} - x - 1 \qquad \underline{\underline{f''(0) = 0}}$

$f'''(x) = e^{x} - 1$

x		0	
f'''	−	0	+
f''	↘ +	0	+ ↗
f'	↗ −	0	+ ↗
f	↘		↗

$\longrightarrow e^{x} - 1 = 0 \Rightarrow e^{x} = 1 \Leftrightarrow x = 0.$

20. ⓐ $f(x)=e^{x-1}-\ln x \qquad D_f=(0,+\infty)$

$f'(x)=e^{x-1}-\frac{1}{x} \qquad f'(1)=0$

$f''(x)=e^{x-1}+\frac{1}{x^2}>0.$

x	0		1		+∞
f''		+		+	
f'		↗ −	0	↗ +	
f		↘		↗	

$x<1 \Rightarrow f'(x)<f'(1) \Rightarrow f'(x)<0$

$x>1 \Rightarrow f'(x)>f'(1) \Rightarrow f'(x)>0$

ⓑ $f(x)=(x-1)\ln x \qquad D_f=(0,+\infty).$

$f'(x)=\ln x+(x-1)\frac{1}{x} = \ln x+1-\frac{1}{x}$

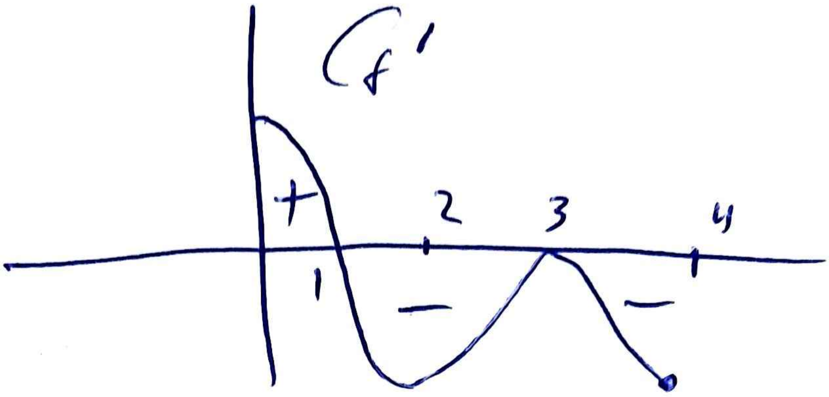
$$\boxed{f'(x)=\ln x+1-\frac{1}{x}}$$

$f'(1)=0$

$f''(x)=\frac{1}{x}+\frac{1}{x^2}>0.$

x	0		1		+∞
f''		+		+	
f'		↗ −	0	+ ↗	
f		↘		↗.	

17.



x		1	
f'	+	0	−
f	↗		↘

19. $f'(2)=0$

$f''(x)<0 \quad \forall x \in \mathbb{R}$.

x		2	
f''	−		−
f'	+ ↘	0	− ↘
f	↗		↘

$x<2 \Rightarrow f'(x)>f'(2) \Rightarrow f'(x)>0$

$x>2 \Rightarrow f'(x)<f'(2) \Rightarrow f'(x)<0$

10. (ε) $f(x) = e^x + \frac{x^2}{2} - 2x + \text{ημ}x$

$f'(x) = e^x + x - 2 + \text{συν}x$, $\underline{f'(0) = 0}$

$f''(x) = \underset{\oplus}{e^x} + \underbrace{1 - \text{ημ}x}_{\oplus} > 0$

• $\text{ημ}x < 1 \Rightarrow 1 - \text{ημ}x > 0$

x		0	
f''	+		+
f'	↗ −	0	+ ↗
f	↘		↗

21. (α) $f(x) = x - \frac{\ln x}{x}$ $\qquad D_f = (0, +\infty)$

$f'(x) = 1 - \frac{\frac{1}{x}x - \ln x}{x^2}$

$f'(x) = 1 - \frac{1 - \ln x}{x^2}$

$$\boxed{f'(x) = \frac{x^2 - 1 + \ln x}{x^2}}$$

Θετω $g(x) = x^2 - 1 + \ln x$ $g(1) = 0$.

$$g'(x) = 2x + \frac{1}{x} > 0$$

x	0	1 +∞
g'	+	+
g	↗ −	0 + ↗
f'	−	+
f	↘	↗

η g(x) ειναι ο αριθμητης της f'(x) και εχουν το ιδιο προσημο γιατι ο παρονομαστης της f(x) ειναι θετικος και δεν παιζει ρολο στα προσημα.

1. $f(x) = e^x + x^3 + x$

2. $f(x) = x^3 - 6x^2 - 15x + 7$

3. $f(x) = \frac{x^2+4}{x}$

4. $f(x) = 2\eta\mu x - x$, $x \in [-\pi, \pi]$.

5. $f(x) = x^2(2\ln x - 5) - 4x(\ln x - 3)$

6. $f(x) = e^x(x^2 - 4x + 5)$

7. $f(x) = x + \sigma\upsilon\nu x - 5$

8. $f(x) = -2x^2 + 4x - (x-1)\ln x$

9. $f(x) = 2e^x - x^2 - x$

10. $f(x) = 6x^2 \ln x - 2x^3 - \frac{7}{2}x^2 + 7x$

11. $f(x) = \dfrac{\ln x}{x-2}$

12. $f(x) = \begin{cases} 2x^3 - 3x^2, & x \leq 1 \\ -2x^3 + 9x^2 - 12x + 4, & x > 1 \end{cases}$

13. $f(x) = \begin{cases} x^2 - 2x, & x < 3 \\ e^x(x-3) + 2, & x \geq 3 \end{cases}$

14. $f(x) = \begin{cases} \frac{1}{x} - \ln x, & 0 < x \leq 1 \\ 2 - x - x^4, & x > 1 \end{cases}$

15. $f(x) = \ln |x| + \dfrac{1}{2x^2}$

16. $f(x) = x^x, \ x > 0$

17. $f(x) = e^x - \dfrac{x^2}{2} - x - 1$

18. $f(x) = 2x \ln x - x^2$

19. $f(x) = e^{-x} + \ln x$

20. $f(x)=e^x-\frac{1}{2}x^2+\sigma\upsilon\nu x$

ⓐ $f(x) = e^x + x^3 + x$ $D_f = R$

$f'(x) = e^x + 3x^2 + 1 > 0$ αρα f ↗ στο R.

και δεν εχει ακροτατα.

ⓑ $f(x) = x^3 - 6x^2 - 15x + 7.$ $D_f = R.$

$f'(x) = 3x^2 - 12x - 15$

$f'(x) = 0 \Leftrightarrow 3x^2 - 12x - 15 = 0 \Leftrightarrow x^2 - 4x - 5 = 0$

$\boxed{x = 5}$ $\boxed{x = -1}$

x		-1		5	
f'(x)	+	0	-	0	+
f(x)	↗		↘		↗

Η f(x) γν. αυξουσα στο $(-\infty, -1]$ και $[5, +\infty)$

Η f(x) γν. φθινουσα στο $[-1, 5]$.

Το $A(-1, 15)$ ειναι τοπικο μεγιστο

Το $B(5, -93)$ ειναι τοπικο ελαχιστο.

(γ) $f(x)=\frac{x^2+4}{x}$ $\qquad D_f=\mathbb{R}^*$

$$f'(x)=\frac{2x\cdot x-(x^2+4)}{x^2}=\frac{2x^2-x^2-4}{x^2}=\frac{x^2-4}{x^2}$$

$$f'(x)=0 \Leftrightarrow \frac{x^2-4}{x^2}=0 \Leftrightarrow x^2-4=0 \Leftrightarrow \boxed{x=2} \text{ ή } \boxed{x=-2}$$

x		-2		0		2	
x^2-4	+	0	-	▨	-	0	+
x^2	+		+	▨	+		+
$f'(x)$	+		-	▨	-		+
$f(x)$	↗		↘	▨	↘		↗

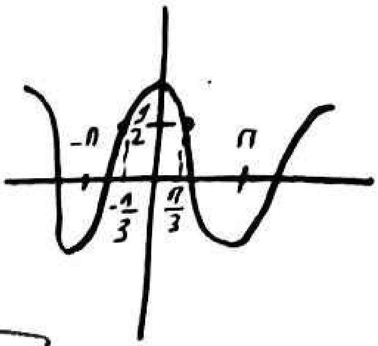
Η f(x) ειναι γνησιως αυξουσα στο $(-\infty,-2]$ και $[2,+\infty)$

Η f(x) ειναι γνησιως φθινουσα στο $[-2,0)$ και $(0,2]$.

Το A(-2, -4) ειναι τοπικο μεγιστο

Το B(2, 4) ειναι τοπικο ελαχιστο.

(δ) $f(x) = 2ημx - x$, $x \in [-π, π]$.



$f'(x) = 2συνx - 1$.

$f'(x) = 0 \iff 2συνx - 1 = 0 \iff συνx = \frac{1}{2}$

$\boxed{x = \frac{π}{3}}$ ή $\boxed{x = -\frac{π}{3}}$

x	−π		$-\frac{π}{3}$		$\frac{π}{3}$		π
f'(x)		−	0	+	0	−	
f(x)		↘		↗		↘	

Αντιπροσωποι

$f'(π) = 2συνπ - 1 = -3$

$f'(-π) = 2συν(-π) - 1 = -3$

$f'(0) = 2συν0 - 1 = 1$

Η f(x) γν. φθινουσα στο $[-π, -\frac{π}{3}]$ και $[\frac{π}{3}, π]$.

Η f(x) γν. αυξουσα στο $[-\frac{π}{3}, \frac{π}{3}]$

Το $A(-π, -π)$ ειναι τοπικο μεγιστο.

Το $B(-\frac{π}{3}, -\sqrt{3} + \frac{π}{3})$ ειναι τοπικο ελαχιστο.

Το $Γ(\frac{π}{3}, \sqrt{3} - \frac{π}{3})$ ειναι τοπικο μεγιστο

Το $Δ(π, -π)$ ειναι τοπικο ελαχιστο.

(ε) $f(x)= x^2(2\ln x-5)-4x(\ln x-3)$ $\quad D_f=(0,+\infty)$.

$$f'(x)= 2x(2\ln x-5)+x^2\cdot\frac{2}{x}-4(\ln x-3)-4x\,\frac{1}{x}$$

$$f'(x)= 4x\ln x-10x+2x-4\ln x+12-4$$

$$f'(x)=4x\ln x-4\ln x-8x+8$$

$$f'(x)=4\ln x(x-1)-8(x-1)$$

$$f'(x)=4(x-1)(\ln x-2).$$

$f'(x)=0 \iff 4(x-1)(\ln x-2)=0$

$x-1=0$ ή $\ln x-2=0$

$\boxed{x=1}$

$\ln x=2$

$\boxed{x=e^2}$

x	0		1		e^2		$+\infty$
$4(x-1)$		−	0	+		+	
$\ln x-2$		−		−	0	+	
$f'(x)$		+		−		+	
$f(x)$		↗		↘		↗	

Η $f(x)$ γν. αυξουσα στο $(0,1]$ και $[e^2,+\infty)$

Η $f(x)$ γν. φθινουσα στο $[1,e^2]$

Το $A(1,\ 7)$. ειναι τοπικο μεγιστο.

Το $B(e^2,\ 4e^2-e^4)$ ειναι τοπικο ελαχιστο.

(ζ) $f(x) = e^x(x^2-4x+5).$ $D_f = R$

$f'(x) = e^x(x^2-4x+5) + e^x(2x-4)$

$f'(x) = e^x(x^2-4x+5+2x-4)$

$f'(x) = e^x(x^2-2x+1) = e^x(x-1)^2 \geq 0$

Αρα η $f(x)$ γνησιω/ αυξουσα στο R.

(η) $f(x) = x + συνx - 5$, $D_f = R$.

$f'(x) = 1 - ημx \geq 0$ αρα f γν. αυξουσα στο R.

Γνωριζω οτι $ημx \leq 1 \Rightarrow 1 - ημx \geq 0$

(θ). $f(x) = -2x^2+4x-(x-1)\ln x$ $D_f = (0,+\infty)$

$f'(x) = -4x+4 - \ln x - \frac{x-1}{x}$

$f'(x) = -4x+4 - \ln x - 1 + \frac{1}{x}$

$f'(x) = -4x - \ln x + \frac{1}{x} + 3.$ Παρατηρω οτι $f'(1) = 0$

$f''(x) = -4 - \frac{1}{x} - \frac{1}{x^2} < 0$

---

x	0		1		+∞
$f''(x)$		−		−	
$f'(x)$		↘ +	0	− ↘	
$f(x)$		↗		↘	

$x<1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x) > 0$

$x>1 \Rightarrow f'(x) < f'(1) \Rightarrow f'(x) < 0$

Η $f(x)$ γν. αυξουσα στο $(0, 1]$

Η $f(x)$ γν. φθινουσα στο $[1, +\infty)$.

Το $A(1, 2e)$. ειναι ολικο μεγιστο.

ⓘ $f(x) = 2e^x - x^2 - x$ $\qquad D_f = \mathbb{R}.$

$f'(x) = 2e^x - 2x - 1.$

$f''(x) = 2e^x - 2$

Λυνω την εξισωση $f''(x) = 0$ $\Leftrightarrow 2e^x - 2 = 0$

$e^x = 1$

$\boxed{x = 0}$

x		0	
$f''(x)$	$-$	0	$+$
$f'(x)$	↘ $+$		$+$ ↗
$f(x)$	↗		↗

$x < 0 \Rightarrow f'(x) > f'(0) \Leftrightarrow f'(x) > 1$

$x > 0 \Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 1$

(Ε) $f(x) = 6x^2 \ln x - 2x^3 - \frac{7}{2}x^2 + 7x$ $\quad D_f = (0, +\infty)$

$f'(x) = 12x \ln x + \frac{6x^2}{x} - 6x^2 - \frac{7}{2}2x + 7$

$f'(x) = 12x \ln x + 6x - 6x^2 - 7x + 7$

$f'(x) = 12x \ln x - 6x^2 - x + 7$

$f''(x) = 12 \ln x + 12 - 12x - 1$

$f''(x) = 12 \ln x - 12x + 11$

$f'''(x) = \frac{12}{x} - 12$ $\quad$ Παρατηρω οτι $f'''(1) = 0$

$f^{(IV)}(x) = -\frac{12}{x^2} < 0$.

x	0		1		$+\infty$
f''''		−		−	
f'''		↘ +	0	− ↘	
f''		↗ −		↘ −	
f'		↘ +	0	− ↘	
f		↗		↘	

$x < 1 \Rightarrow f'''(x) > f'''(1) \Rightarrow f'''(x) > 0$

$x > 1 \Rightarrow f'''(x) < f'''(1) \Rightarrow f'''(x) < 0$.

$x < 1 \Rightarrow f''(x) < f''(1) \Rightarrow f''(x) < -1$

$x > 1 \Rightarrow f''(x) < f''(1) \Rightarrow f''(x) < -1$

$x < 1 \Rightarrow f'(x) > f'(1) \Rightarrow f'(x) > 0$

$x > 1 \Rightarrow f'(x) < f'(1) \Rightarrow f'(x) < 0$.

① $f(x) = \dfrac{\ln x}{x-2}$ $D_f = (0,2) \cup (2,+\infty)$.

$$f'(x) = \frac{\frac{1}{x}(x-2) - \ln x}{(x-2)^2} = \frac{x-2-x\ln x}{x(x-2)^2} = \frac{\varphi(x)}{x(x-2)^2}$$

Θεωρω $\varphi(x) = x-2-x\ln x$.

$\varphi'(x) = 1 - \ln x - 1 = -\ln x$

Λυνω $\varphi'(x)=0 \Leftrightarrow -\ln x = 0 \Leftrightarrow x = 1$.

x	0	1	2
$\varphi'(x)$	+	0 −	−
$\varphi(x)$	↗ −	↘ −	↘ −
$x(x-2)^2$	+	+	+
$f'(x)$	−	−	−
$f(x)$	↘	↘	↘

$x<1 \Rightarrow \varphi(x) < \varphi(1) \Rightarrow \varphi(x) < -1$

$x>1 \Rightarrow \varphi(x) < \varphi(1) \Rightarrow \varphi(x) < -1$

Η $f(x)$ γν. φθινουσα στο $(0,2)$ και $(2,+\infty)$

και δεν εχει ακροτατα.

(p) $f(x) = \begin{cases} 2x^3 - 3x^2, & x \leq 1 \\ -2x^3 + 9x^2 - 12x + 4, & x > 1 \end{cases}$

- εξεταζω αν η $f(x)$ ειναι συνεχης στο 1.
- Δεν με ενδιαφερει αν η $f(x)$ ειναι παραγ/μη στο 1,

$$\left.\begin{matrix} \lim\limits_{x\to1^-} f(x) = -1 \\ \lim\limits_{x\to1^+} f(x) = -1 \end{matrix}\right\} \lim\limits_{x\to1} f(x) = -1 = f(1)$$ Ειναι συνεχη στο 1.

1. Αν $x \in (-\infty, 1)$ η $f_1'(x) = 6x^2 - 6x$

Λυνω $f_1'(x) = 0 \Leftrightarrow 6x^2 - 6x = 0 \Leftrightarrow x(x-1) = 0$

$\boxed{x=0}$ $\boxed{x=1}$

2. Αν $x \in (1, +\infty)$ η $f_2'(x) = -6x^2 + 18x - 12$

Λυνω $f_2'(x) = 0 \Leftrightarrow -6x^2 + 18x - 12 = 0$

$x^2 - 3x + 2 = 0$

$\boxed{x=1}$ $\boxed{x=2}$

---

x		0		1		2	
$f_1'(x)$	+	0	−	0	+		+
$f_2'(x)$	+		+	0	−	0	+
$f'(x)$	+		−		−		+
$f(x)$	↗		↘		↘		↗

Η $f(x)$ γν. αυξουσα στο $(-\infty, 0]$ και $[2, +\infty)$

Η $f(x)$ γν. φθινουσα στο $[0, 2]$

Το $A(0, f(0))$ Τοπικο Μεγιστο

Το $B(2, f(2))$ Τοπικο Ελαχιστο.

ⓥ $f(x) = \begin{cases} x^2-2x, & x<3 \\ e^x(x-3)+2, & x\geq 3 \end{cases}$

- $\lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^-} \left[x^2-2x\right] = 3$
- $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} \left[e^x(x-3)+2\right] = 2$

$\Big\}$ Το οριο $\lim\limits_{x\to3} f(x)$ δεν υπαρχει.

$\underline{x\in(-\infty,3)}$

$f_1(x) = x^2-2x$

$f_1'(x) = 2x-2$

$\boxed{x=1}$

$\underline{x\in[3,+\infty)}$

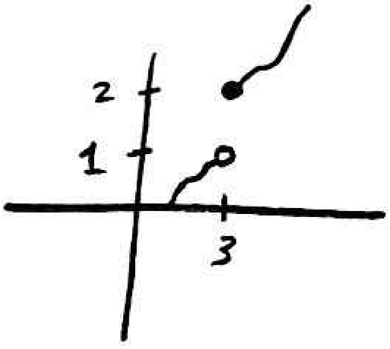
$f_2(x) = e^x(x-3)+2$

$f_2'(x) = e^x(x-3)+e^x$

$f_2'(x) = e^x(x-2)$

$\boxed{x=2}$

x		1	2	3	
$f_1'(x)$	− 0	+	+	▨	+
$f_2'(x)$	−	− 0	+	▨	+
$f'(x)$	−	+	+	▨	+
$f(x)$	↘	↗	↗	▨	↗



Η $f(x)$ γν. φθινουσα $(-\infty,1]$

Η $f(x)$ γν. αυξουσα $[1,+\infty)$.

Το $A(1,f(1))$ Τ.Ε.

③ $f(x) = \begin{cases} \frac{1}{x} - \ln x, & 0 < x \le 1 \\ 2 - x - x^4, & x > 1 \end{cases}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left[\frac{1}{x} - \ln x\right] = 1$

$\lim_{x \to 1^+} f(x) = \underset{x \to 1^+}{\ln} \left[2 - x - x^4\right] = 0$

} Το οριο $\lim_{x \to 1} f(x)$ δεν υπαρχει.

$x \in (0, 1]$

$f_1(x) = \frac{1}{x} - \ln x$

$f_2'(x) = -\frac{1}{x^2} - \frac{1}{x} < 0$

αφου $x > 0$

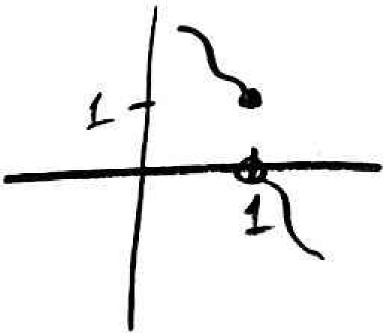
$x \in (1, +\infty)$

$f_2(x) = 2 - x - x^4$

$f_2'(x) = -1 - 4x^3 < 0$

αφου $x > 0$.

x	0	1
$f_1'(x)$	−	−
$f_2'(x)$	−	−
$f'(x)$	−	−
$f(x)$	↘	↘



Η $f(x)$ γν. φθινουσα στο $(0, +\infty)$.

(ο) $f(x) = \ln|x| + \frac{1}{2x^2}$

Πρεπει $|x| > 0$ αρα $x \neq 0$ συνεπως $D_f = \mathbb{R}^*$

$$f'(x) = \frac{1}{x} - \frac{2x}{2x^4} = \frac{1}{x} - \frac{1}{x^3} = \frac{x^2-1}{x^3}$$

$\rightarrow f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Leftrightarrow x = 1$ ή $x = -1$.

x		-1		0		1	
x^2-1	$+$	0	$-$	‖	$-$	0	$+$
x^3	$-$		$-$	0	$+$		$+$
$f'(x)$	$-$		$+$	‖	$-$		$+$
$f(x)$	↘		↗	‖	↘		↗

(π) $f(x) = x^x \quad , x > 0$

$$f(x) = x^x = e^{\ln x^x} = e^{x\ln x}$$

$$f'(x) = e^{x\ln x}\,(x\ln x)' = x^x(\ln x + 1)$$

$\rightarrow f'(x) = 0 \Leftrightarrow \ln x + 1 = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = \frac{1}{e}$.

x	0		$\frac{1}{e}$		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$		↘		↗	

(σ) $f(x) = 2x\ln x - x^2$ $\qquad D_f = (0,+\infty)$

$f'(x) = 2\ln x + 2 - 2x = 2(\ln x + 1 - x) \leq 0$

Γνωστη ανισοτητα

$\ln x \leq x - 1$

$\Leftrightarrow f \searrow$

(τ) $f(x) = e^{-x} + \ln x$, $D_f = (0,+\infty)$

$f'(x) = -e^{-x} + \frac{1}{x} = -\frac{1}{e^x} + \frac{1}{x} = \frac{e^x - x}{xe^x} > 0 \quad f \nearrow$

Επειδη $e^x \geq x+1 > x \Leftrightarrow e^x > x \Leftrightarrow e^x - x > 0$

(υ) $f(x) = -\sigma\varphi x + \frac{1}{x}$, $x \in (0,\pi)$.

$f'(x) = \frac{1}{\eta\mu^2 x} - \frac{1}{x^2} = \frac{x^2 - \eta\mu^2 x}{x^2\eta\mu^2 x} > 0 \Rightarrow f \nearrow$

Γνωστη ανισοτητα

$|\eta\mu x| \leq |x|$

Αν $x \neq 0$ τοτε $|\eta\mu x| < |x| \Leftrightarrow |\eta\mu x|^2 < |x|^2$

$\Leftrightarrow \eta\mu^2 x < x^2 \Rightarrow x^2 - \eta\mu^2 x > 0$

(φ) $f(x) = x - \sqrt{x^2+1}$, $D_f = R$

$f'(x) = 1 - \frac{2x}{2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1} - x}{\sqrt{x^2+1}}$

→ $f'(x) = 0 \Leftrightarrow \sqrt{x^2+1} - x = 0 \Leftrightarrow \sqrt{x^2+1} = x$

1. Αν $x \geq 0$ τοτε $\sqrt{x^2+1}^2 = x^2 \Leftrightarrow x^2 + 1 = x^2$

$1 = 0$

Αδυνατη.

2. Αν $x < 0$ τοτε αδυνατη.

Συνεπως $f'(x) \neq 0$ αρα $f'(x) > 0$ ή $f'(x) < 0$ $\forall x \in R$.

Επειδη $f'(0) = 1 \Rightarrow f'(x) > 0$ $\forall x \in R$

Αρα $f \nearrow$.

Σελ 16

2. $f(x) = \frac{e^x}{x}$ $\quad \Delta = [\ln 2, \ln 4]$,

Η f παρ/μη στο $(\ln 2, \ln 4)$ ως π.π.σ

Η f συνεχ/ή στο $[\ln 2, \ln 4]$ ως π.σ.σ.

$$f(\ln 2) = \frac{e^{\ln 2}}{\ln 2} = \frac{2}{\ln 2}$$

$$f(\ln 4) = \frac{e^{\ln 4}}{\ln 4} = \frac{4}{\ln 2^2} = \frac{4}{2\ln 2} = \frac{2}{\ln 2}$$

$f(\ln 2) = f(\ln 4)$

Rolle $\exists \xi \in (\ln 2, \ln 4)$ τ.ω $f'(\xi) = 0$.

$$f'(x) = \frac{e^x x - e^x}{x^2}$$

$f'(x) = 0 \Rightarrow e^x x - e^x = 0 \Rightarrow e^x(x-1) = 0$.

$\boxed{x=1}$.

18. $f'(x) \neq 0 \quad \forall x \in R$.

ⓐ Εστω οτι η $f(x)$ δεν ειναι 1-1.

Αρα θα υπαρχουν $f(x_1) = f(x_2)$

Rolle

$f'(\xi) = 0$

Ατοπο

γιατι $f'(x) \neq 0$.

Αρα η f 1-1

ⓑ. $f(ημx) - f(x) = 0$

$f(ημx) = f(x)$

f 1-1

$ημx = x$

$$\boxed{x = 0}$$

⑧. $A(1,3)$ $f(1)=3$

$B(-2,9)$ $f(-2)=9$.

$f^{-1}(f(x)-6)=1$

$f(x)-6=f(1)$

$f(x)-6=3$

$f(x)=9$

$f(x)=f(-2)$

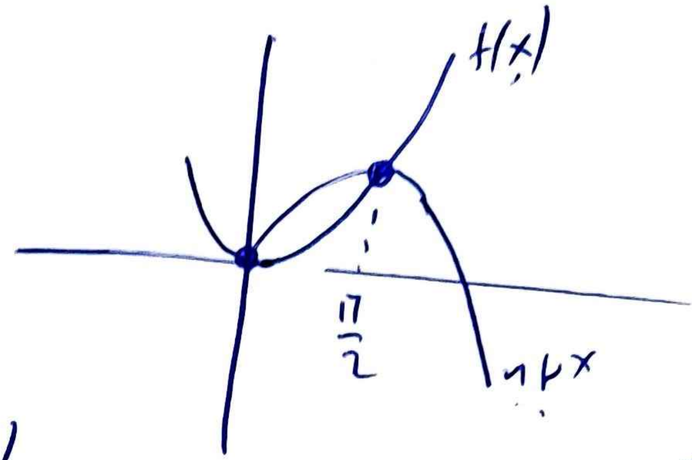
$f\,9)=1$

$x=-2$

17. ⓐ νδο $\exists \xi \in (0, \frac{\pi}{2})$

τ.ω οι εφαπτομενες

των C_f, C_g στα



$A(\xi, f(\xi))$ και $B(\xi, g(\xi))$ να ειναι παραλληλες!

Αρκει νδο $\exists \xi \in (0, \frac{\pi}{2})$ τ.ω $g'(\xi) = f'(\xi)$.

$$g'(x) = f'(x)$$

$$συνx = f'(x)$$

$$\boxed{\underbrace{συνx - f'(x)}_{h(x)} = 0}$$

$$\underbrace{ημx - f(x)}_{H(x)} = 0$$

$$\left.\begin{matrix} H(0) = -f(0) = 0 \\ H(\frac{\pi}{2}) = 1 - 1 = 0 \end{matrix}\right\} \text{Rolle } H'(\xi) = 0$$

$$h(\xi) = 0.$$

$$συν\xi - f'(\xi) = 0$$ ✓

Ⓑ $f'(x) = συνx$ $\quad (0, \frac{π}{2})$

$συνx - f'(x) = 0$

$h(x) = 0$.

από πριν $h(ξ) = 0$ ✓

15. $f(x)=3^x+x^2-3x-1$

(α) $f(0)=1-1=0$

$f(1)=3+1-3-1=0$

$f(0)=f(1)$

Η f παρ/μη στο $(0,1)$ ως π.π.σ.

Η f συνεχής στο $[0,1]$ ως π.σ.σ.

(β) Έστω ότι η f έχει τρείς ρίζες

$$f(x_1)=f(x_2)=f(x_3)=0$$

$$\underbrace{\quad\quad}_{\text{Rolle}}\ \underbrace{\quad\quad}_{\text{Rolle}}$$

$$\underbrace{f'(\xi_1)=0 \quad f'(\xi_2)=0}_{\text{Rolle}}$$

$$f''(\xi)=0$$

$f'(x)=3^x\ln 3+2x-3$

$f''(x)=\ln 3\, 3^x \ln 3+2>0$

άρα $f''(x)=0$ Αδύνατο!

Άρα έχει το πολύ 2 ρίζες.

(γ) . $g(x) = 3^x$ $h(x) = -x^2+3x+1$.

$g(x) = h(x)$

$3^x = -x^2+3x+1$

$3^x + x^2 - 3x + 1 = 0$

$f(x) = 0$

$f(0) = 0$ $f(1) = 0$.

Εχει τουλαχιστον δυο ριζ.

πριν εδειξα οτι εχει το

πολυ 2.

Αρα 2 ακριβως

$x = 0$ $x = 1$

13. $f'(x) \neq -1 \quad \forall x \in R$.

Νδο η $f(x)$ και $\varepsilon : y = -x$ εχουν το πολυ ενα κοινο σημειο.

$f(x) = -x$

$\underbrace{f(x) + x}_{g(x)} = 0$

Αρκει νδο η $g(x)$ εχει το πολυ μια ριζα.

Εστω οτι η $g(x)$ εχει δυο ριζ

$g(x_1) = g(x_2) = 0$.

Rolle $\exists\, \xi \in (x_1, x_2)$ τ.ω $g'(\xi) = 0$

$g'(x) = f'(x) + 1$

$g'(\xi) = f'(\xi) + 1 = 0 \Rightarrow f'(\xi) = -1$ Ατοπο.

Αρα ειναι αδυνατον η

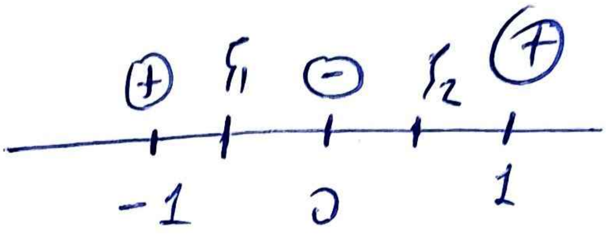
g(x) να εχει δυο ριζες

αρα θα εχει το πολυ

μια .

11. (α) $2x^4+\lambda^2(x^2-1)=x$ $\quad \lambda\neq 0$.

Νδο εχει δυο τουλ. ριζα $(-1,1)$.



$$\boxed{f(x)=2x^4+\lambda^2(x^2-1)-x}$$

$f(-1)=2+1=3$

$f(0)=-\lambda^2$

$f(1)=1$

$f(-1)f(0)<0$

Bolzano $\exists\, \xi_1\in(-1,0)$

τ.ω $f(\xi_1)=0$

$f(0)f(1)<0$

Bolzano $\exists\, \xi_2\in(0,1)$

τ.ω $f(\xi_2)=0$ ..

Β' τροπος

$F(x)=\frac{2}{5}x^5+\frac{\lambda^2}{3}x^3-\lambda^2x-\frac{1}{2}x^2$.

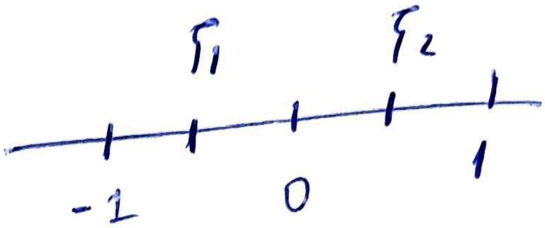
$F(-1)=-\frac{2}{5}-\frac{\lambda^2}{3}+\lambda^2-\frac{1}{2}$

$F(0)=0$

ΟΧΙ $=$

δεν ισχυει ο Rolle για την παραγουσα

Ⓑ $8x^3+2\lambda^2 x = 1$



$f(\xi_1) = f(\xi_2) = 0$

Rolle

$\exists\, \xi \in (\xi_1, \xi_2)$

τ.ω $f'(\xi) = 0$.

$f'(x) = 8x^3 + 2\lambda^2 x - 1$

$8\xi^3 + 2\lambda^2 \xi - 1 = 0$

$$\boxed{8\xi^3 + 2\lambda^2 \xi = 1}$$

9. $f(x) > 0 \quad \forall x \in [0, a]$

$$f(a) = e\,f(0)$$

$g(x) = \ln f(x) - a x$ Ισχυει ο Rolle στο $[0, \alpha]$.

$$\Rightarrow g(0) = g(\alpha).$$

ⓐ $\ln f(0) - \alpha \cdot 0 = \ln f(a) - a^2$

$$\ln \frac{f(a)}{e} = \ln f(a) - a^2$$

$$\cancel{\ln f(a)} - \ln e = \cancel{\ln f(a)} - a^2$$

$$-1 = -a^2$$

$$\boxed{\alpha = 1} \qquad \cancel{\boxed{a = -1}}$$

(Β) Αφου ισχυει ο Rolle για την $g(x)$ στο $[0, a]$ $\exists\, \xi$ τ.ω $g'(\xi) = 0$.

$$g'(x) = \frac{f'(x)}{f(x)} - a \longrightarrow \frac{f'(\xi)}{f(\xi)} - a = 0 \longleftarrow$$

$$f'(\xi) = a f(\xi)$$

$$\boxed{a = 1}$$

8. $g(x) = e^{x} f(x)$

$f(0) = f\left(\frac{3}{2}\right) = 0$

Ν.δ.ο $\exists \xi \in \left(0, \frac{3}{2}\right)$ τ.ω $f'(\xi) = -f(\xi)$

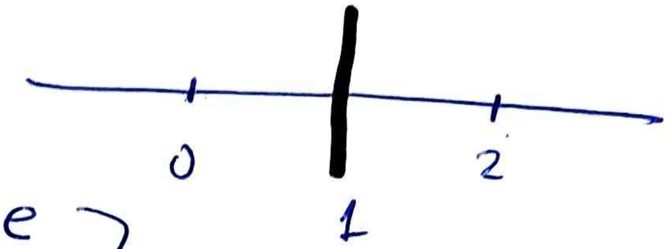
$$\left.\begin{array}{l} g(0) = e^{0} f(0) = 0 \\ g\left(\frac{3}{2}\right) = e^{3/2} f\left(\frac{3}{2}\right) = 0 \end{array}\right\} \begin{array}{l} \text{Rolle} \\ \exists \xi \in \left(0, \frac{3}{2}\right) \\ \text{τ.ω } g'(\xi) = 0 \end{array}$$

$$g'(x) = e^{x} f(x) + e^{x} f'(x)$$

$$g'(\xi) = \cancel{e^{\xi}} f(\xi) + \cancel{e^{\xi}} f'(\xi) = 0$$

$$f(\xi) + f'(\xi) = 0$$

3. (β) $f(x) = \begin{cases} e^x , & x<1 \\ x-1 , & x \geq 1 \end{cases}$ $\Delta = [0,2]$



$$\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} e^x = e \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x-1 = 0 \end{array}\right\}$$ Όχι συνεχής στο 1.

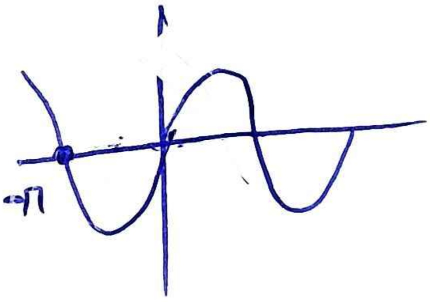
Δεν ικανοποιείται ο Rolle στο [0,2].

4. $f(x) = (x+1)ημx$

(α) $\boxed{f'(x) = ημx + (x+1)συνx}$.

$f'(-1) = ημ(-1) < 0$

$f'(0) = 1 > 0$



$\exists\, ξ \in (-1,0)$ τ.ω $f'(ξ) = 0$.

β' τρόπος

$f(-1)=0$
$f(0)=0$ $\}$ Rolle $\exists \xi \in (-1,0)$

τ.ω $f'(\xi)=0$.

(B) ν.δ.ο η εξίσωση $εφx = -x-1$ έχει στο $(-1,0)$

Έχω ήδη δείξει ότι $\exists \xi \in (-1,0)$

τ.ω $f'(\xi)=0$.

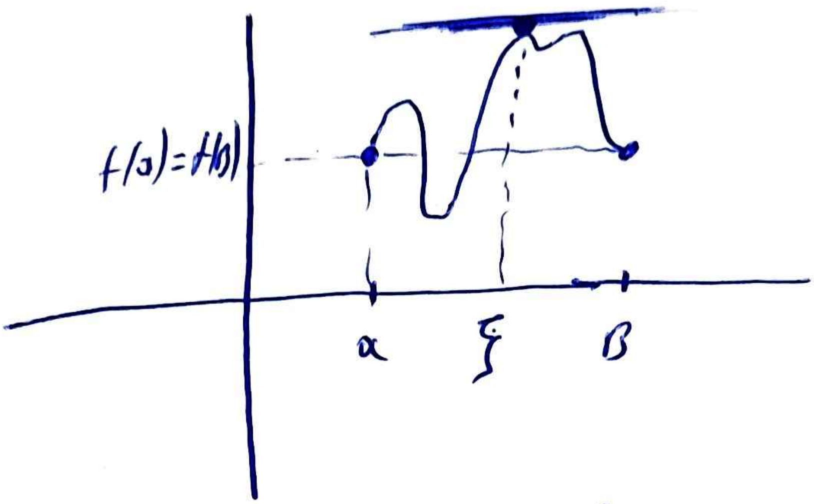
$ημ\xi + (\xi+1)\,συν\xi = 0$.

$$\frac{ημ\xi}{συν\xi} + \xi + 1 = 0$$

$εφ\xi + \xi + 1 = 0$

$εφ\xi = -\xi - 1$ ✓

Θεωρημα Rolle



$\varepsilon: y - f(\xi) = f'(\xi)(x-\xi)$

↓

$f'(\xi) = 0$

- f παρ/μη (α, β) .
- f συνεχής $[\alpha, \beta]$
- $f(\alpha) = f(\beta)$

Τότε $\exists\, \xi \in (\alpha, \beta)$ τ.ω $f'(\xi) = 0$

Επομω Μαθημ

Σελ 16

2 α

3 α

5

6

7

10

12

14

16

19

20

21

22

23

24

Σελ. 16

2. ⓐ $f(x) = x^2 + 2x + 1$. $\Delta = [-2, 0]$

Η f(x) συνεχής $[-2, 0]$ ως π.σ.σ

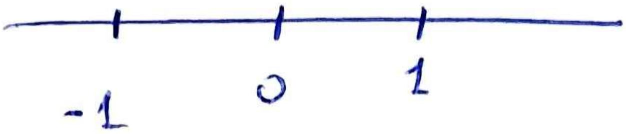
Η f(x) παρ/μη στο $(-2, 0)$ ως π.π.σ.

$$\left.\begin{array}{l} f(-2) = 1 \\ f(0) = 1 \end{array}\right\} f(0) = f(-2)$$ Rolle ✓

$\rightarrow f'(x) = 0 \Rightarrow 2x + 2 = 0 \Leftrightarrow \underline{\underline{x = -1}}$

$f'(x) = 2x + 2$

3. ⓐ $f(x)=\begin{cases} x^2 & , x<0 \\ x^3 & , x\geq 0 \end{cases}$



Bolle ok!

$\left.\begin{array}{l} f(-1)=1 \\ f(1)=1 \end{array}\right\} f(-1)=f(1)$

$\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} x^2=0 \\ \lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x^3=0 \end{array}\right\} \lim\limits_{x\to 0} f(x)=f(0)$

Συνεχής στο 0!

Επίσης συνεχής $[-1,0)$ και $(0,1]$ ως π.σ.σ.

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^{\cancel{2}}-0}{\cancel{x}}=0$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^{\cancel{3}}}{\cancel{x}}=0$$

Παρ/μη στο 0.

Παρ/μη στο $(-1,0)$ και $(0,1)$ ως π.π.σ.

5. $f(x) = x^4 - x^3 + \lambda x^2 - \lambda x.$

ⓐ $\left.\begin{array}{l} f(0)=0 \\ f(1)=0 \end{array}\right\} f(0)=f(1)$ ✓

Η f συνεχής ως πολυωνυμική στο $[0,1]$

Η f παρ/μη ως πολυωνυμική στο $(0,1)$

ⓑ ·Επειδή ισχύει ο Rolle για την $f(x)$ στο $[0,1]$ $\exists \xi \in (0,1)$ τ.ω $f'(\xi)=0$.

$f'(x) = 4x^3 - 3x^2 + 2\lambda x - \lambda$

$f'(\xi) = 4\xi^3 - 3\xi^2 + 2\lambda\xi - \lambda = 0$

6. $f(x) = συνx \cdot \ln x$

(α) $y - f(ξ) = f'(ξ)(x - ξ) \ // \ x'x$

$f'(ξ) = 0.$

Αρκει νδο $\exists ξ \in (1, \frac{π}{2})$ τ.ω

$f(1) = συν1 \ln 1 = 0$

$f(\frac{π}{2}) = συν\frac{π}{2} \ln(\frac{π}{2}) = 0.$

$f(1) = f(\frac{π}{2})$ Rolle $\exists ξ \in (1, \frac{π}{2})$ τ.ω $\underline{f'(ξ)=0.}$

(β) $f'(x) = -ημx \ln x + \frac{συνx}{x}$

$f'(ξ)=0 \Rightarrow -ημξ \ln ξ + \frac{συνξ}{ξ} = 0$

$-ξ ημξ \ln ξ + συνξ = 0.$

$-ξ \ln ξ + \frac{συνξ}{ημξ} = 0$

$$\boxed{σφξ = ξ \ln ξ.}$$

12. $f'(x) \neq 3x^2$

$H(x)=x^3 \Rightarrow f(x)-x^3=0 \Rightarrow h(x)=f(x)-x^3.$

Εστω οτι εχει δυο ριζ

$$\underbrace{h(x_1)=0=h(x_2)}_{\text{Rolle}}$$

$\exists \xi \in (x_1,x_2)$ τ.ω $h'(\xi)=0.$

$h'(x)=f'(x)-3x^2$

$h'(\xi)=f'(\xi)-3\xi^2=0$

$f'(\xi)=3\xi^2$ Ατοπο!

Αρα εχει το πολυ μια ριζα.

14. $f(0)=0$ $\qquad$ $f(1)=1$

$f''(x) \neq 2 \quad \forall x \in \mathbb{R}.$

ⓐ $g(x)=f(x)-x^2$ $\qquad$ $[0,1]$.

$$\left.\begin{array}{l} g(0)=f(0)=0 \\ g(1)=f(1)-1=1-1=0 \end{array}\right\} g(0)=g(1)$$

Η g συνεχής στο $[0,1]$ ως π.σ.σ

Η g παρ/μη στο $(0,1)$ ως π.π.σ.

ⓑ. $f(x)=x^2 \Rightarrow f(x)-x^2=0 \Rightarrow h(x)=0$

Εστω οτι εχει τρεις ριζ.

$$\underbrace{h(x_1)=h(x_2)}\!\!\!\underbrace{=h(x_3)}=0$$

$$\underbrace{h'(\xi_1)=h'(\xi_2)}=0$$

$$h''(\xi)=0$$

$h'(x)=f'(x)-2x$

$h''(x)=f''(x)-2$

$h''(\xi)=f''(\xi)-2=0 \Rightarrow f''(\xi)=2$

Ατοπο!

Αρα εχει το πολυ δυο ριζ.

(γ) $f(x) = x^2 \Rightarrow f(x) - x^2 = 0$

$h(x) = 0.$

$h(0) = f(0) - 0 = 0$

$h(1) = f(1) - 1 = 0.$

Το 0, 1 ρίζα και αφού

έχει το πολύ 2 έχει

ακριβώς 2.

7. ⓐ $g(x) = f(x)\,\eta\mu x$

$g(0) = f(0)\cdot \eta\mu 0 = 0$

$g(\pi) = f(\pi)\,\eta\mu\pi = 0$

$\Big\}\ g(0)=g(\pi)$ Rolle ✓

ⓑ $\exists\, \xi \in (0,\pi)$ τ.ω. $g'(\xi) = 0$.

$g'(x) = f'(x)\,\eta\mu x + f(x)\,\sigma\upsilon\nu x$.

$f'(\xi)\,\eta\mu\xi + f(\xi)\,\sigma\upsilon\nu\xi = 0$.

$$f'(\xi) + f(\xi)\frac{\sigma\upsilon\nu\xi}{\eta\mu\xi} = 0$$

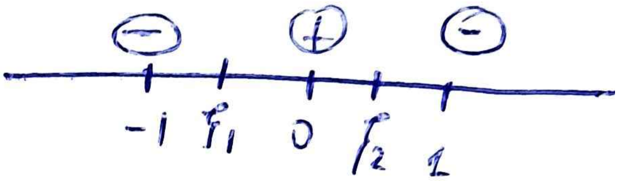
$$\boxed{f'(\xi) = -f(\xi)\,\sigma\varphi\xi}$$

10. $f(x)=x^4-10x^3-15x^2-x+1.$

ⓐ. $\dfrac{10x^3+15x^2+x}{x^4+1}=1 \iff 10x^3+15x^2+x=x^4+1$

$x^4-10x^3-15x^2-x+1=0$

$f(x)=0.$



$f(-1)=1+10-15+1+1=-1$

$f(0)=1$

$f(1)=-24$

$f(-1)\cdot f(0)<0$ Bolzano $\exists\, \xi_1 \in (-1,0)$ τ.ω $f(\xi_1)=0$

$f(0)\, f(1)<0$ Bolzano $\exists\, \xi_2 \in (0,1)$ τ.ω $f(\xi_2)=0$

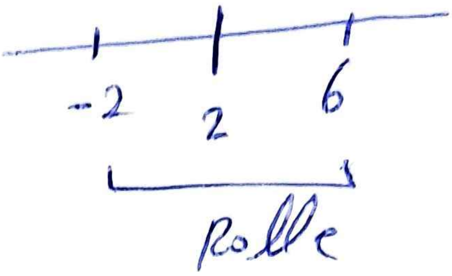
ⓑ $\underbrace{f(\xi_1)=f(\xi_2)=0}_{\text{Rolle}}$

$f'(\xi)=0.$

$f'(x)=4x^3-30x^2-30x-1.$

$f'(\xi)=4\xi^3-30\xi^2-30\xi-1=0.$

16. ⓐ $f(x) = \begin{cases} x^2-6x+3, & x>2 \\ -2x-1, & x\leq 2 \end{cases}$.



$$\left.\begin{array}{l} f(-2)=3 \\ f(6)=3 \end{array}\right\} f(-2)=f(6).$$

$$\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = -5 \\ \lim\limits_{x\to 2^+} f(x) = -5 \end{array}\right\} \lim\limits_{x\to 2} f(x)=f(2)$$ ✓

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{-2x-1+5}{x-2} =$$

$$= \lim_{x\to 2^-} \frac{-2\cancel{(x-2)}}{\cancel{x-2}} = \boxed{-2}$$ ✓

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{x^2-6x+3+5}{x-2} =$$

$$= \lim_{x\to 2^+} \frac{x^2-6x+8}{x-2} = \lim_{x\to 2} 2x-6 = \boxed{-2}$$

$f'(x)=0$

$x \leq 2$

$f'(x)=0.$

$-2=0$

Ατοπο.

$x > 2$

$f'(x)=0.$

$2x-6=0$

$x=3$

Ⓑ $f(x)=1+ημ3x$

$f(0)=1$

$f(π)=1+ημ3π=1$ } ✓

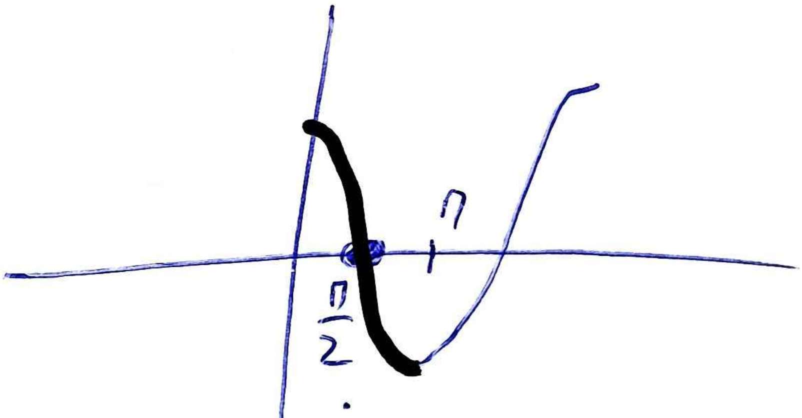
Συνεχης και παρ/μη ?

$f'(x)=0.$

$3συν3x=0$

$συν3x=0$

$3x=\frac{π}{2}$



$x=\frac{π}{6}$

20. $f'(x) \neq 3x^2$

$2 < f(x) < 3$

Άρα δεν υπάρχει φ και σχν δύο ρίζ θα σχν το πολύ μια.

$f(x_0) = x_0^3 + 2$

$$\underbrace{f(x) - x^3 - 2}_{\varphi(x)} = 0$$

$$\left.\begin{array}{l} \varphi(0) = f(0) - 2 > 0 \\ \varphi(1) = f(1) - 3 < 0 \end{array}\right\} \varphi(0)\varphi(1) < 0$$

$\exists\, x_0 \in (0,1)$ τ.ω $\varphi(x_0) = 0$ ✓

$f(x_0) = x_0^3 + 2$

Έστω ότι υπάρχει επιπλέον μια x_1 τ.ω

$$\underbrace{\varphi(x_1) = \varphi(x_0) = 0}_{\text{Rolle}}$$

$\varphi'(\xi) = 0$ $\quad \varphi'(x) = f'(x) - 3x^2$

$\varphi'(\xi) = f'(\xi) - 3\xi^2 \Rightarrow f'(\xi) = 3\xi^2$ άτοπο

19. $f'(x) \neq 0 \quad \forall x \in R$.

$f(x) = -f(2-x) \quad \forall x \in R$.

Εξίσωση $f(x) = 0$ ..

Αφού $f'(x) \neq 0 \Rightarrow f$ 31-1

$f(x) = f(1)$

f 31-1

$x = 1$

Σελ 19

21. Εχω οτι $\alpha < \beta$ και $\frac{\alpha}{\beta} = e^{\alpha-\beta}$

(α) νδο $f(x) = \frac{x}{e^x}$ ισχυει ο Rolle $[\alpha,\beta]$.

$f(\alpha) = \frac{\alpha}{e^{\alpha}}$

$f(\beta) = \frac{\beta}{e^{\beta}}$

Γνωριζω οτι $\frac{\alpha}{\beta} = e^{\alpha-\beta} \Leftrightarrow \frac{\alpha}{\beta} = \frac{e^{\alpha}}{e^{\beta}}$

$\Leftrightarrow \frac{\alpha}{e^{\alpha}} = \frac{\beta}{e^{\beta}} \Leftrightarrow f(\alpha) = f(\beta)$

Η f συνεχης στο $[\alpha,\beta]$ ως π.δ.σ

Η f παρ/μη στο (α,β) ως π.π.δ.

(β) $f'(\xi) = 0$

$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x}$

$f'(x) = 0$

$\boxed{x=1}$

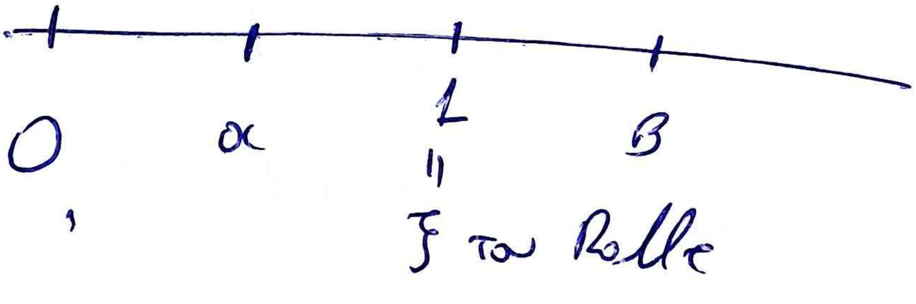
Γνωρίζω ότι

$$\frac{a}{\underset{\oplus}{\beta}} = \frac{\overset{\oplus}{e^{a}}}{\underset{\oplus}{e^{\beta}}}$$

Προκύπτει ότι $a>0$.

Ο Rolle μου εξασφαλίσε ότι η $f'(x)$ έχει μια τουλάχιστον ρίζα στο $[a,\beta]$.

$f'(x) = \frac{1-x}{e^{x}}$ $\Leftrightarrow f'(x)=0$ $\underline{\underline{x=1}}$



$0 < a < 1 < \beta$.

22. $\alpha < \beta$ και $\dfrac{\sigma\upsilon\nu\alpha - \sigma\upsilon\nu\beta}{\beta - \alpha} = \dfrac{\beta+\alpha}{2}$

νδο $\underline{\underline{\alpha\beta < 0}}$

$$2(\sigma\upsilon\nu\alpha - \sigma\upsilon\nu\beta) = \beta^2 - \alpha^2$$

$$2\sigma\upsilon\nu\alpha - 2\sigma\upsilon\nu\beta = \beta^2 - \alpha^2$$

$$\boxed{\alpha^2 + 2\sigma\upsilon\nu\alpha = \beta^2 + 2\sigma\upsilon\nu\beta}$$

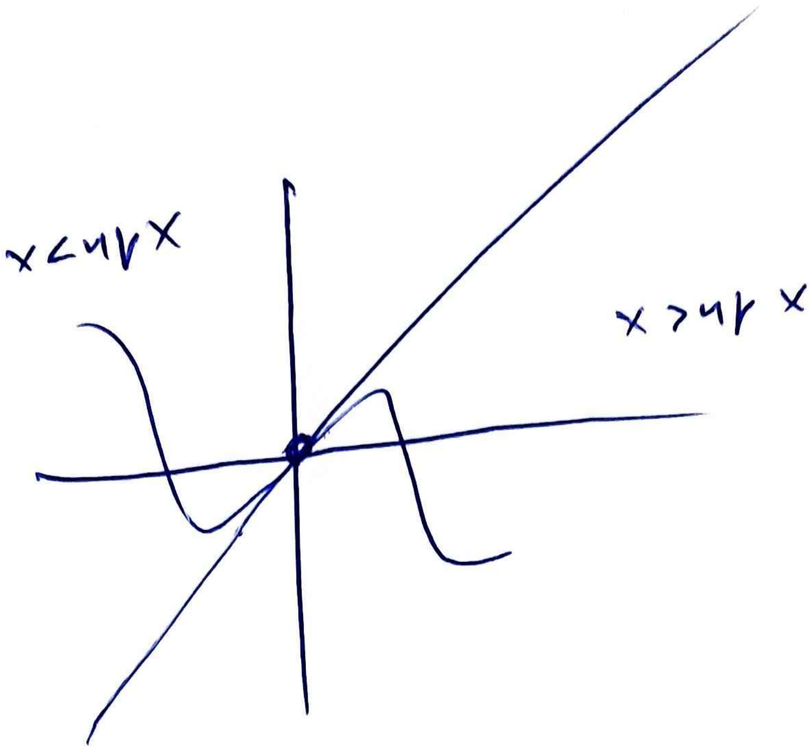
$$f(x) = x^2 + 2\sigma\upsilon\nu x$$

$$f(\alpha) = f(\beta)$$

Rolle $\exists \xi \in (\alpha, \beta)$ τω $f'(\xi) = 0$

$$f'(x) = 2x - 2\eta\mu x$$

$$f'(\xi) = 2\xi - 2\eta\mu\xi = 0 \Rightarrow \xi = \cancel{\eta\mu}\,\xi \quad \boxed{\xi = 0}$$



$$\alpha < \xi < \beta$$

$$\alpha < 0 < \beta$$

$$\alpha\beta < 0$$

23.

x		2	3	
$f(x)$	.	0	0	

ⓐ $G(x)=\frac{f(x)}{x-4}$

$$\left.\begin{array}{l} G(2)=\frac{f(2)}{2-4}=0 \\ \\ G(3)=\frac{f(3)}{3-4}=0 \end{array}\right\} \begin{array}{l} G(2)=G(3) \quad \text{Rolle} \\ \exists\xi\in(2,3) \ \text{τ.ω} \ G'(\xi)=0 \end{array}$$

$$G'(x)=\frac{f'(x)(x-4)-f(x)}{(x-4)^2}$$

$$G'(\xi)=\frac{f'(\xi)(\xi-4)-f(\xi)}{(\xi-4)^2}=0$$

$$\boxed{f'(\xi)(\xi-4)-f(\xi)=0.}$$

ⓑ $y-f(\xi)=f'(\xi)(x-\xi) \longrightarrow A(4,0)$

$0-f(\xi)=f'(\xi)(4-\xi)$ ✓

24. $3f(2) = 2f(3)$

ⓐ $\frac{f(2)}{2} = \frac{f(3)}{3}$ $\boxed{\varphi(x) = \frac{f(x)}{x}}$

$\varphi(2) = \varphi(3)$

Rolle $\exists\, \xi \in (2,3)$ τω $\underline{\varphi'(\xi) = 0}$

$$\varphi'(x) = \frac{f'(x)x - f(x)}{x^2}$$

$$\varphi'(\xi) = \frac{f'(\xi)\xi - f(\xi)}{\xi^2} = 0$$

$\underline{\xi f'(\xi) - f(\xi) = 0.}$

$$\boxed{f'(\xi) = \frac{f(\xi)}{\xi}}$$

ⓑ $\cdot\ y - f(x_0) = f'(x_0)(x - x_0) \longrightarrow (0,0)$

$0 - f(x_0) = f'(x_0)(0 - x_0)$

$-f(x_0) = -x_0 f'(x_0) \Rightarrow f'(x_0) = \frac{f(x_0)}{x_0}$ ✓

3. $f(x) = x - \ln(x^2+1)$

ⓐ $f'(x) = 1 - \frac{2x}{x^2+1} = \frac{x^2+1-2x}{x^2+1}$

$f'(x) = \frac{(x-1)^2}{x^2+1} \geqslant 0$ $\quad f \nearrow$.

Ⓑ i). $x = \ln(x^2+1)$

$x - \ln(x^2+1) = 0$

$f(x) = 0$

$f(x) = f(0)$

$f\ 1-1$

$\boxed{x=0}$

ii). $f(x^2+1) > f(e^x)$

$f \nearrow$

$x^2+1 > e^x$

$\ln(x^2+1) > x$ $\quad \Rightarrow \quad 0 > x - \ln(x^2+1)$

$0 > f(x)$

$f(0) > f(x)$ $\quad \underline{\underline{x<0}}$

iii) $e^{f(x)} - 1 = f^2(x)$

$$e^{f(x)} = f^2(x) + 1$$

$$f(x) = \ln(f^2(x)+1)$$

$$f(x) - \ln(f^2(x)+1) = 0$$

$$f(f(x)) = f(0)$$

f g1-1

$$f(x) = 0$$

$$f(x) = f(0)$$

f g1-1

$$\boxed{x=0}$$

(γ) νδο $\ln(n^2+1) + f(\ln n) < n$

$$f(\ln n) < n - \ln(n^2+1)$$

$$f(\ln n) < f(n)$$

$$f \nearrow$$

$\ln n < n$.

Αφου $\ln x \le x-1 \Rightarrow \ln x < x \Rightarrow \ln n < n$ ✓

(δ) $$\lim_{x\to 0} \frac{\ln x}{f(3x)-f(2x)} = \lim_{x\to 0^+} \frac{\ln x}{f(3x)-f(2x)}$$

$$= \lim_{x\to 0^+} \ln x \cdot \frac{1}{\underset{\oplus}{f(3x)-f(2x)}} = -\infty \cdot (+\infty)$$

$$= -\infty$$

$2x < 3x$

$f \nearrow$

$f(2x) < f(3x)$

$$\boxed{0 < f(3x)-f(2x)}$$

(ε). Αν $f(\alpha) < 0$ νδο $e^{\alpha} < 1$

$f(\alpha) < 0$

$f(\alpha) < f(0)$

$f \nearrow$

$\alpha < 0$

$e^{\alpha} < e^{0}$

$$\underline{e^{\alpha} < 1}$$

2. $f(x) = x^7 + 2x - 3$

$f'(x) = 7x^6 + 2 > 0$

$f \nearrow$

$f\ 3\ 1-1$.

ⓐ $x^7 + 2x = 3$

$x^7 + 2x - 3 = 0$

$f(x) = 0$

$f(x) = f(1)$

$f\ 3\ 1-1$

$\boxed{x = 1}$

ⓑ .i) $x < \frac{3}{x^6+2}$ $\Rightarrow$ $x(x^6+2) < 3$

$x^7 + 2x - 3 < 0$

$f(x) < 0$

$f(x) < f(1)$

$f \nearrow$

$\underline{x < 1}$

11). $e^{7x}+2e^{x}>3$

$e^{7x}+2e^{x}-3>0$

$f(e^{x})>f(1)$

$f \nearrow$

$e^{x}>1$ $\qquad$ $\underline{\underline{x>0}}$

(γ). i) $f(e^{x}-1)-2x=x^{7}-3$

$f(e^{x}-1)=x^{7}+2x-3$

$f(e^{x}-1)=f(x)$

$f : 1-1$

$e^{x}-1=x$

$e^{x}-x-1=0$

$\underline{\underline{x=0}}$

Βασικη ανισοτητα

$e^{x}\geq x+1$

$x=0$

iii. $f(x^7+x-3)+x^7=-2x-3$

$f(x^7+x-3) = -x^7-2x-3$

$f(x^7+x-3) = f(-x)$

$f\ \overset{o}{\text{ɔ}}\ 1-1$

$x^7+x-3=-x$

$x^7+2x-3=0$

$f(x)=0$

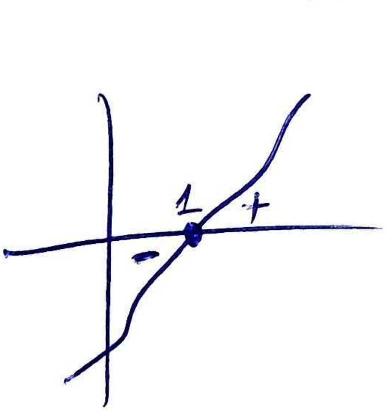
$f(x)=f(1)$

$\boxed{x=1}$

(δ) $\lim\limits_{x\to 1} \dfrac{1}{f(x)}$

$\lim\limits_{x\to 1^-} \dfrac{1}{f(x)} = -\infty$

$\lim\limits_{x\to 1^+} \dfrac{1}{f(x)} = +\infty$



Το όριο δεν υπάρχει.

Σελ 94

1. $f(x)=x^7+5x-6$.

(α) $f'(x)=7x^6+5>0$

$f\nearrow$

(β) $3<\pi \Rightarrow f(3)<f(\pi)$.

(γ) i) $1<x<2$

$f\nearrow$

$f(1)<f(x)<f(2)$

$$\boxed{0<f(x)}$$

ii). Αν $\alpha>\beta \xrightarrow{f\nearrow} f(\alpha)>f(\beta)$

$$\Rightarrow \alpha^7+5\alpha-\cancel{6}>\beta^7+5\beta-\cancel{6}$$

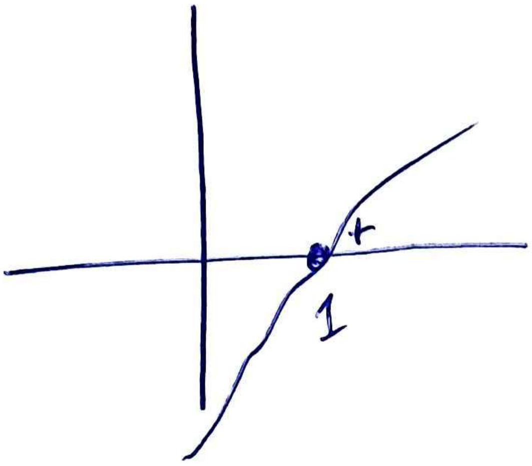
$$\alpha^7-\beta^7>5\beta-5\alpha$$

$$\alpha^7-\beta^7>5(\beta-\alpha)$$

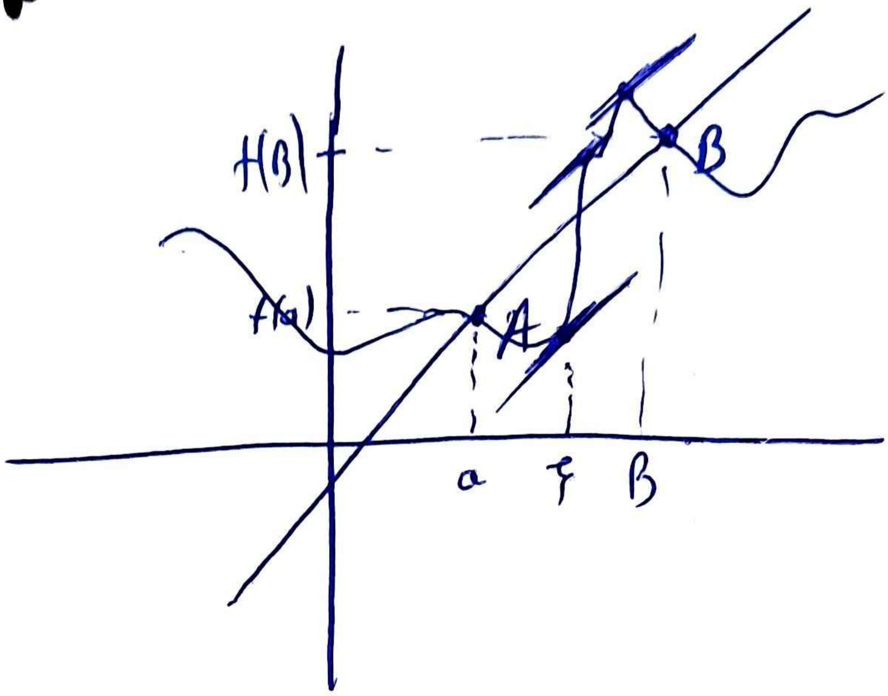
$$\alpha^7-\beta^7>-5(\alpha-\beta)$$

$$\boxed{\frac{\alpha^7-\beta^7}{\alpha-\beta}>-5}.$$

⑤ $\lim_{x \to 1^+} \frac{1}{\underset{\oplus}{f(x)}} = +\infty$



Θεώρημα Μέσης Τιμής



f συνεχής στο [α, β]

f παρ/μη στο (α, β).

$$f'(\xi) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}.$$

$$\boxed{\lambda_{AB} = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}}$$

$\varepsilon_{AB} \parallel y - f(\xi) = f'(\xi)(x-\xi)$

2024 Γ' Θεμα

$$f(x) = \begin{cases} -2x+4+e^{\lambda}, & 0 \leq x < 2 \\ -x^2+4x-3+\lambda, & x \geq 2 \end{cases}$$ Συνεχής

Γ1 Νδο $\lambda = 0$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-2x+4+e^{\lambda}) = e^{\lambda}$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x^2+4x-3+\lambda) = 1+\lambda$$

Πρέπει $e^{\lambda} = \lambda + 1 \iff \underline{\underline{\lambda = 0}}$

Αφού $e^x \geq x+1$ με το "=" να ισχύει για $x=0$

$$f(x) = \begin{cases} -2x+5, & 0 \leq x < 2 \\ -x^2+4x-3, & x \geq 2. \end{cases}$$

Γ2

$0 \leq x < 2$

$f_1(x) = -2x+5$

$f_1'(x) = -2 < 0$

$f_1 \downarrow$

$x \geq 2$

$f_2(x) = -x^2+4x-3$

$f_2'(x) = -2x+4$

$\rightarrow -2x+4 = 0$

$2x = 4$

$\boxed{x=2}$.

x	0	2	$+\infty$
f_1'	$-$		▨ $-$ ▨
f_2'	▨ $+$ ▨	0	$-$
f'	$-$		$-$
f	$\searrow$		$\searrow$

και δεν εχει ακροτατα,

Γ3 i) Η f συνεχης στο [0,3] ως π.σ.σ.

$$\bullet \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{-2x+5-1}{x-2} = \lim_{x\to 2^-} \frac{-2x+4}{x-2}$$

$$= \lim_{x\to 2^-} \frac{-2}{1} = \boxed{-2}$$

$$\bullet \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{-x^2+4x-3-1}{x-2} = \lim_{x\to 2} \frac{-x^2+4x-4}{x-2}$$

$$= \lim_{x \to 2} \frac{-2x+4}{1} = 0 .$$

Δεν είναι παρ/μη στο 2.

Άρα δεν ικανοποιείται το ΘΜΤ

στο $[0,3]$.

ii).

$$y - f(\xi) = f'(\xi)(x - \xi) \quad // \quad \varepsilon_{\Delta E}.$$

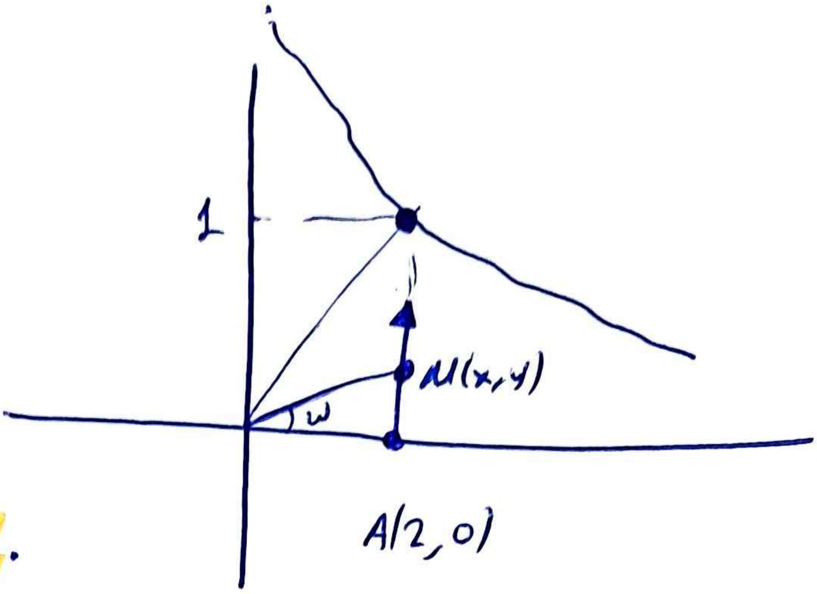
$$f'(\xi) = \lambda_{\Delta E} = \frac{f(3) - f(0)}{3 - 0}.$$

$$f'(\xi) = \frac{0 - 5}{3} \quad \Leftrightarrow f'(\xi) = \frac{-5}{3}$$

$$f'(x) = -\frac{5}{3}$$

$0 \le x < 2$	$x \geq 2$
$f'(x) = -\frac{5}{3}$	$f'(x) = -\frac{5}{3}.$
$-2 = -\frac{5}{3}$	$-2x+4 = -\frac{5}{3}$
Άτοπο	$-6x+12 = -5$
	$-6x = -17$
$\boxed{\xi = \frac{17}{6}}$	$\boxed{x = \frac{17}{6}.}$

Γ4



Η ταχυτητα ειναι
ο ρυθμος μεταβολης
της θεσης

Μου δινει $y'(t)=\frac{1}{2}$.

$x(t)=2$.

$y(t_1)=1$

$$εφ\hat{ω} = \frac{y}{x}$$

$$εφ\,ω(t) = \frac{y(t)}{x(t)}$$

$$\frac{1}{συν^2ω(t)}\,ω'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t)}.$$

$$ω'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t)} \quad συν^2ω(t)$$

$$ω'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t)} \quad \frac{1}{1+εφ^2ω(t)}$$

$$w'(t) = \frac{y'(t)\,x(t) - y(t)\,x'(t)}{x^2(t)} \cdot \frac{1}{1+\left(\frac{y(t)}{x(t)}\right)^2}$$

$t = t_1$

$$w'(t_1) = \frac{y'(t_1)\,x(t_1) - y(t_1)\,x'(t_1)}{x^2(t_1)} \cdot \frac{1}{1+\left(\frac{y(t_1)}{x(t_1)}\right)^2}$$

$$w'(t_1) = \frac{\frac{1}{2}2 - 1 \cdot 0}{4} \cdot \frac{1}{1+\left(\frac{1}{2}\right)^2}$$

$$w'(t_1) = \frac{1}{4} \cdot \frac{1}{1+\frac{1}{4}} = \frac{1}{\cancel{4}} \cdot \frac{1}{\frac{5}{\cancel{4}}} = \frac{1}{5}$$

$$\boxed{w'(t_1) = \frac{1}{5}}$$

4. Σελ 32

$f(1) < f(0)$

ⓐ νδο $\exists \xi \in (0,1)$ τ.ω $f'(\xi) < 0$

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0) < 0$$

ⓑ. Αν $f'(x) \neq 0$ $\forall x \in \mathbb{R}$.

Αφού $f'(x) \neq 0$ και συνεχής

Είναι συνεχής γιατί f είναι

2 φορές παρ/μη είναι

η f' συνεχής

Άρα $f'(x) > 0$ ή $f'(x) < 0$ $\forall x \in \mathbb{R}$

$f'(\xi) < 0$ $\quad$ $f'(x) \underset{\forall x}{<} 0$

5. ⑧ NSo $\exists \xi \in (x, x+1)$

T.w $f(x) + f'(\xi) = f(x+1)$.

$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1 - x}$$

$$f'(\xi) = \underline{f(x+1) - f(x)}$$

$$\boxed{f'(\xi) + f(x) = f(x+1)}$$

Επομενο Μαθημα

Σελ 32

②

③

⑤ α β γ.

Αυριο.

Για τριτη

Σελ 95

④

⑤

⑥

⑦

⑧

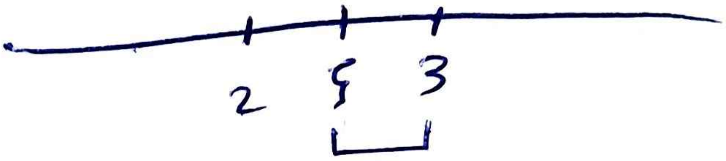
⑨

⑩

10. $f' \downarrow$

ⓐ צ"ל $f'(3) < f(3)$ אם $f(2) = 0$.

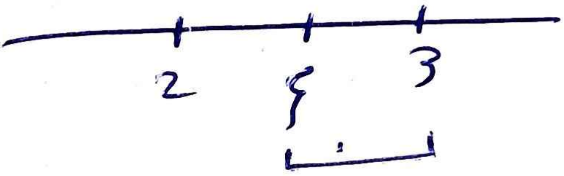
$f'(\xi) = \frac{f(3)-f(2)}{3-2} = f(3)$



$\xi < 3 \Rightarrow f'(\xi) > f'(3) \Rightarrow \underline{f(3) > f'(3)}$

ⓑ צ"ל $f'(3) + f(2) < f(3)$.

$f'(\xi) = \frac{f(3)-f(2)}{3-2}$



$\boxed{f'(\xi) = f(3) - f(2)}$

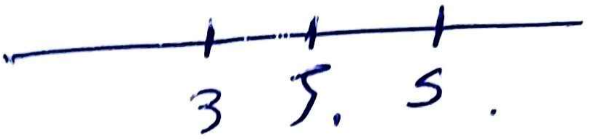
$\xi < 3 \xrightarrow{f' \downarrow} f'(\xi) > f'(3)$

$f(3) - f(2) > f'(3)$

$\underline{f(3) > f(2) + f'(3)}$

⑧ vdo $f(5) < f(3) + 2f'(3)$.

$$f'(\xi) = \frac{f(5) - f(3)}{5 - 3}$$



$$\boxed{f'(5) = \frac{f(5) - f(3)}{2}}$$

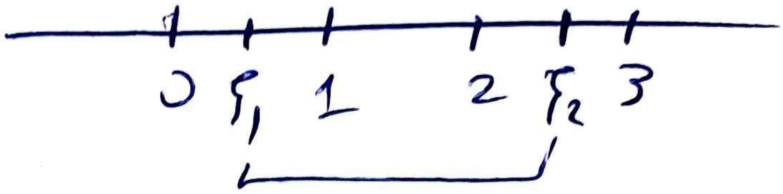
$$3 < \xi \overset{f'\downarrow}{\Rightarrow} f'(3) > f'(\xi)$$

$$f'(3) > \frac{f(5) - f(3)}{2}$$

$$2f'(3) > f(5) - f(3).$$

$$\underline{\underline{2f'(3) + f(3) > f(5)}}$$

⑧ νδο $f(1)+f(2) > f(0)+f(3)$



$$f'(\xi_1) = \frac{f(1)-f(0)}{1-0} = f(1)-f(0)$$

$$f'(\xi_2) = \frac{f(3)-f(2)}{3-2} = f(3)-f(2).$$

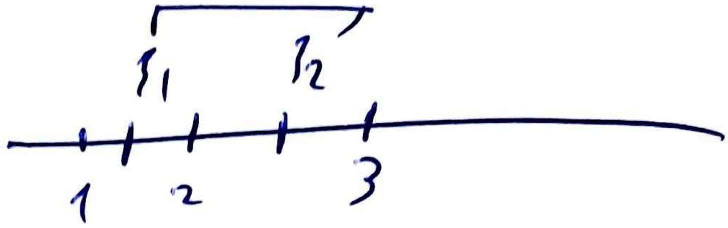
$$\xi_1 < \xi_2 \xrightarrow{f' \downarrow} f'(\xi_1) > f'(\xi_2)$$

$$f(1)-f(0) > f(3)-f(2)$$

$$f(1)+f(2) > f(3)+f(0)$$

✓

(ε) · νδο $f(1)+f(3) < 2f(2)$



$$f'(\xi_1) = \frac{f(2)-f(1)}{2-1} = f(2)-f(1)$$

$$f'(\xi_2) = \frac{f(3)-f(2)}{3-2} = f(3)-f(2).$$

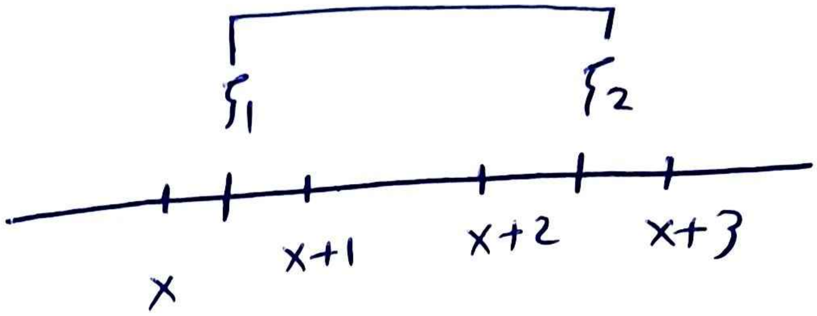
$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) > f'(\xi_2)$$

$$f(2)-f(1) > f(3)-f(2)$$

$$2f(2) > f(3)+f(1)$$

9. $f' \nearrow$

(α) νδο $f(x+1) + f(x+2) < f(x) + f(x+3)$



$$f'(\xi_1) = \frac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$$

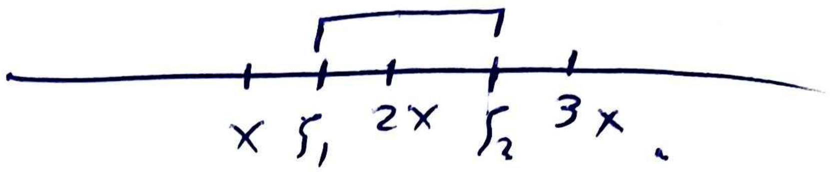
$$f'(\xi_2) = \frac{f(x+3) - f(x+2)}{x+3-(x+2)} = f(x+3) - f(x+2)$$

$$\xi_1 < \xi_2 \xrightarrow{f' \nearrow} f'(\xi_1) < f'(\xi_2)$$

$$f(x+1) - f(x) < f(x+3) - f(x+2)$$

$$\boxed{f(x+1) + f(x+2) < f(x) + f(x+3)}$$

Ⓑ ΝΔΟ $f(x) + f(3x) > 2f(2x)$



$$f'(\xi_1) = \frac{f(2x)-f(x)}{2x-x} = f(2x)-f(x)$$

$$f'(\xi_2) = \frac{f(3x)-f(2x)}{3x-2x} = f(3x)-f(2x).$$

$$\xi_1 < \xi_2 \overset{f'\uparrow}{\Rightarrow} f'(\xi_1) < f'(\xi_2)$$

$$f(2x) - f(x) < f(3x) - f(2x)$$

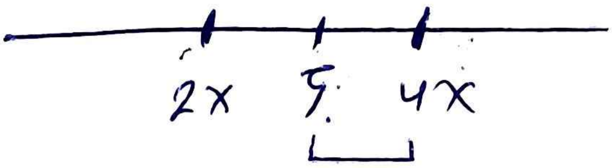
$$2f(2x) < f(3x) + f(x)$$

$$\frac{1}{x} > \frac{\ln(x+1)-\ln x}{x+1-x} > \frac{1}{x+1}$$

8. (α) ιιι). $f' \nearrow$ $\underline{x>0}$

νδο $f(4x) - f(2x) < 2x f'(4x)$.

$$f'(\xi) = \frac{f(4x)-f(2x)}{4x-2x}$$



$$f'(\xi) = \frac{f(4x)-f(2x)}{2x}.$$

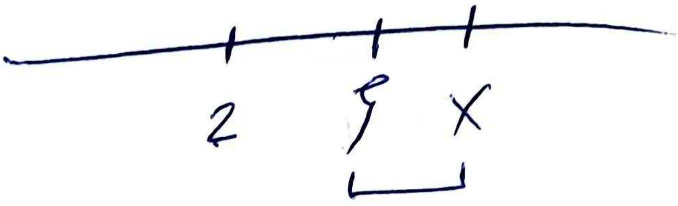
$\xi < 4x \Rightarrow f'(\xi) < f'(4x)$

$$\frac{f(4x)-f(2x)}{2x} < f'(4x)$$

✓ $f(4x) - f(2x) < 2x f'(4x)$. ✓

i) . νδο $f'(x) > \frac{f(x)}{x-2}$ $\forall x>2$ $f(2)=0$

$$f'(\xi) = \frac{f(x)-f(2)}{x-2}$$



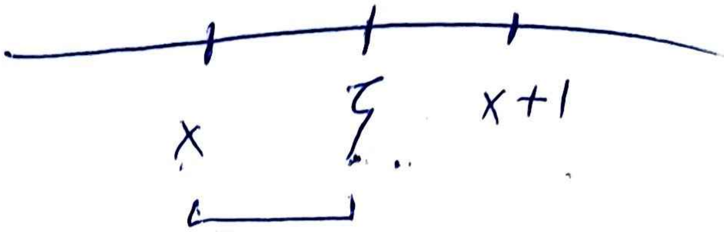
$$\xi < x \xrightarrow{f' \uparrow} f'(\xi) < f'(x)$$

$$\frac{f(x)-f(2)}{x-2} < f'(x)$$

$$\boxed{\frac{f(x)}{x-2} < f'(x)}$$

11). νδο $f'(x) < f(x+1) - f(x)$

$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x}$$



$$\boxed{f'(\xi) = f(x+1) - f(x)}$$

$$x < \xi \xrightarrow{f' \nearrow} f'(x) < f'(\xi)$$

$$f'(x) < f(x+1) - f(x)$$

Ⓑ Ν.δ.ο $\quad ημ2x - ημx > x\,συν2x$

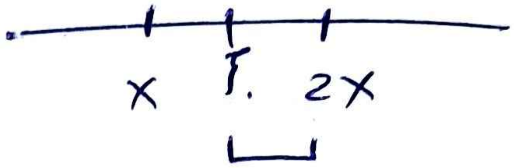
$f''(x) = -\overset{\oplus}{ημx} < 0$

$x \in \left(0, \frac{\pi}{2}\right).$

$f(x) = ημx \qquad f'(x) = συνx$

$f' \downarrow$

$$f'(ξ) = \frac{f(2x) - f(x)}{2x - x}$$



$$f'(ξ) = \frac{ημ2x - ημx}{x}$$

$ξ < 2x$

$f'(ξ) > f'(2x)$

$$\frac{ημ2x - ημx}{x} > συν2x$$

$ημ2x - ημx > x\,συν2x$.

7. νδο $\forall x>0$ $\quad \frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$

$$\frac{1}{x+1} < \ln\left(\frac{x+1}{x}\right) < \frac{1}{x}$$

$$\frac{1}{x+1} < \ln(x+1) - \ln x < \frac{1}{x}$$

$$\boxed{\frac{1}{x+1} < \frac{\ln(x+1) - \ln x}{(x+1) - x} < \frac{1}{x}.}$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$ $\quad f' \downarrow$

$f''(x) = -\frac{1}{x^2} < 0$

$\boxed{\text{ΘΜΤ για την } f \text{ στο } [x, x+1]}$

$$f'(\xi) = \frac{f(x+1) - f(x)}{x+1-x} = \frac{\ln(x+1) - \ln x}{x+1-x}$$

Τελική κίνηση

$x < \xi < x+1 \Rightarrow f'(x) > f'(\xi) > f'(x+1)$

6. (δ) νδο $\frac{1}{3} < \ln\frac{3}{2} < \frac{1}{2}$

$$\frac{1}{3} < \ln 3 - \ln 2 < \frac{1}{2}$$

$$\boxed{\frac{1}{3} < \frac{\ln 3 - \ln 2}{3-2} < \frac{1}{2}}$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$ $\qquad\qquad f' \downarrow$

$f''(x) = -\frac{1}{x^2} < 0$

$$\boxed{\text{ΘΜΤ για τη } f \text{ στο } [2,3]}$$

$$f'(\xi) = \frac{f(3)-f(2)}{3-2} = \frac{\ln 3 - \ln 2}{3-2}.$$

Τελική κίνηση

$$2 < \xi < 3 \xrightarrow{f' \downarrow} f'(2) > f'(\xi) > f'(3)$$

$$\frac{1}{2} > \frac{\ln 3 - \ln 2}{3-2} > \frac{1}{3}$$ ✓

6. (γ) νδο $(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\frac{\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\beta} < (\beta-\alpha)\,\varepsilon\varphi\beta$

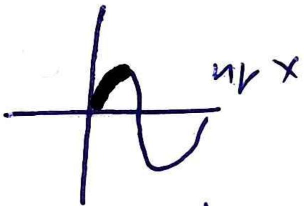
$0 < \alpha < \beta < \frac{\pi}{2}.$

$(\beta-\alpha)\,\varepsilon\varphi\alpha < \ln\sigma\upsilon\nu\alpha - \ln\sigma\upsilon\nu\beta < \beta-\alpha\,\varepsilon\varphi\beta.$

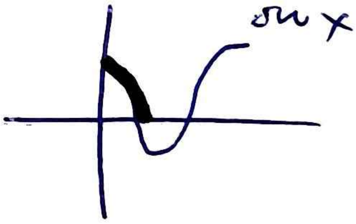
$$\varepsilon\varphi\alpha < \frac{\ln\sigma\upsilon\nu\alpha - \ln\sigma\upsilon\nu\beta}{\beta-\alpha} < \varepsilon\varphi\beta$$

$$\boxed{-\varepsilon\varphi\alpha > \frac{\ln\sigma\upsilon\nu\beta - \ln\sigma\upsilon\nu\alpha}{\beta-\alpha} > -\varepsilon\varphi\beta.}$$

$$\boxed{f(x) = \ln(\sigma\upsilon\nu x)}$$



$$f'(x) = \frac{-\overset{\oplus}{\eta\mu x}}{\underset{\oplus}{\sigma\upsilon\nu x}} = -\varepsilon\varphi x.$$



$f'(x) < 0$ $\quad f \downarrow$

$f''(x) = -\frac{1}{\sigma\upsilon\nu^2 x} < 0$ $\quad f' \downarrow$

Τελικη κινηση

$\alpha < \xi < \beta \xRightarrow{f'} f'(\alpha) > f'(\xi) > f'(\beta)$

5. ⓐ $f'(\xi) = \frac{f(x)-f(1)}{x-1} = \frac{f(x)}{x-1}$.

β) $f'(\xi) = \frac{f(0)-f(x)}{0-x} = \frac{0-f(x)}{-x} = \frac{f(x)}{x}$

$f'(\xi) = \frac{f(x)}{x} \iff \boxed{x f'(\xi) = f(x)}$

γ) $f'(\xi) = \frac{f(2x)-f(x)}{2x-x} = \frac{f(2x)-f(x)}{x}$

$x f'(\xi) = f(2x) - f(x) \Rightarrow \underline{\underline{x f'(\xi) + f(x) = f(2x)}}$

$f'(\alpha) > f'(\xi) > f'(\beta)$

$$-\varepsilon\varphi\alpha > \frac{\ln\sigma\upsilon\nu\beta - \ln\sigma\upsilon\nu\alpha}{\beta-\alpha} > -\varepsilon\varphi\beta \,.$$

✓

ΘΜΤ για την $f(x)$ στο (α,β)

$$f'(\xi) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = \frac{\ln\sigma\upsilon\nu\beta - \ln\sigma\upsilon\nu\alpha}{\beta-\alpha}$$

3. $f(4)-f(1)=9$

ⓐ Νδο $\exists \xi \in (1,4)$ τ.ω $f'(\xi)=3$.

α' τροπος (Bolzano).

$f'(x)=3$

$\underbrace{f'(x)-3}_{g(x)}=0$

Δεν δουλευει!

$g(1)=f'(1)-3$;

β' τροπος (Rolle)

$g(x)=f'(x)-3$.

αρχικη $\boxed{G(x)=f(x)-3x}$

$G(1)=f(1)-3$

$G(4)=f(4)-12=9+f(1)-12=f(1)-3$

$G(1)=G(4)$ Rolle $\exists\exists\xi(1,4)$

τ.ω $G'(\xi)=0$

$g(\xi)=0 \Rightarrow f'(\xi)-3=0$

$f'(\xi)=3$.

γ' τρόπος

$$f'(\xi) = \frac{f(4)-f(1)}{4-1} = \frac{9}{3} = 3.$$

Ⓑ $y - f(\xi) = f'(\xi)(x-\xi) \ /\!/ \ y = 3x - 2.$

Αρκεί να $\exists\ \xi$ τ.ω $f'(\xi) = 3$

✓ Έδειξα πριν ✓

Σελ 32

② (α) $f(x) = x^3 + x$

f συνεχ/ στο $[0, 1]$.

f παρ/μη. στο $[0, 1]$.

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0} = \frac{2 - 0}{1} = 2$$

Άρα $\exists \xi \in (0,1)$ τ.ω $f'(\xi) = 2$.

$f'(x) = 2$

$3x^2 + 1 = 2$

$3x^2 = 1$

$x^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$

$x = \frac{\sqrt{3}}{3}$

Ⓑ $f(x) = \sqrt{x+1}$ $A = [-1,3]$

f συνεχη στο $[-1,3]$

f παρ/μη στο $(-1,3)$.

$\exists\ \xi \in (-1,3)$ τ.ω $f'(\xi) = \dfrac{f(3)-f(-1)}{3-(-1)} = \dfrac{2-0}{4}$

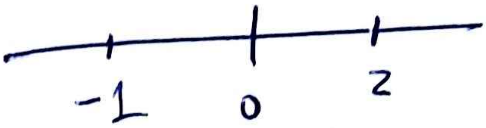
$$f'(\xi) = \frac{1}{2}.$$

$$f'(x) = \frac{1}{2} \Leftrightarrow \frac{1}{2\sqrt{x+1}} = \frac{1}{2} \Leftrightarrow \sqrt{x+1} = 1$$

$$x+1 = 1$$

$$\boxed{x = 0}$$

(γ) $f(x) = \begin{cases} x^2+x, & x<0 \\ x^3+x, & x\geq 0 \end{cases}$ $[-1,2]$.



$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x^2+x) = 0 \\ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (x^3+x) = 0 \end{array}\right\} \begin{array}{l} \lim\limits_{x\to 0} f(x) = 0 \\ f(0) = 0. \end{array}$$

Συνεχής στο 0!.

Άρα συνεχής στο $[-1,0)$ και $(0,2]$ ως π.σ.σ.

Συνεπώς συνεχής $[-1,2]$.

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^2+x-0}{x} = \lim_{x\to 0^-} x+1 = 1$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^3+x}{x} = \lim_{x\to 0^+} x^2+1 = 1.$$

Παρ/μη στο 0!.

Ασκηση

Έστω $f' \nearrow$ και $f(0)=4$

1. Νδο $f(x) < xf'(x)+4$.

2. Νδο $2f(x) < 4 + f(2x)$.

3. Νδο $xf'(0) + f(2x) < f(3x)$.

Επομενο Μαθημα

Ασκηση	Σελ.33	Σελ 95
+	⑥ αβ	④
+	⑭	⑤
	⑲ .	⑥
	㉒	⑦
	㉘	⑧
	㉙ .	⑨
		⑩ .

6. ⓐ $e^{\alpha} < \frac{e^{\beta}-e^{\alpha}}{\beta-\alpha} < e^{\beta}$ $\alpha < \beta$.

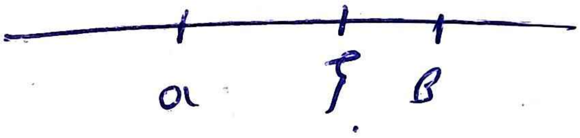
$\boxed{f(x)=e^{x}}$

$f'(x)=e^{x}$ $f' \nearrow$

$f''(x)=e^{x}$

$\boxed{\text{ΘΜΤ για την } f(x) \text{ στο } (\alpha,\beta)}$

$$f'(\xi)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=\frac{e^{\beta}-e^{\alpha}}{\beta-\alpha}.$$



$\alpha < \xi < \beta$

$f' \nearrow$

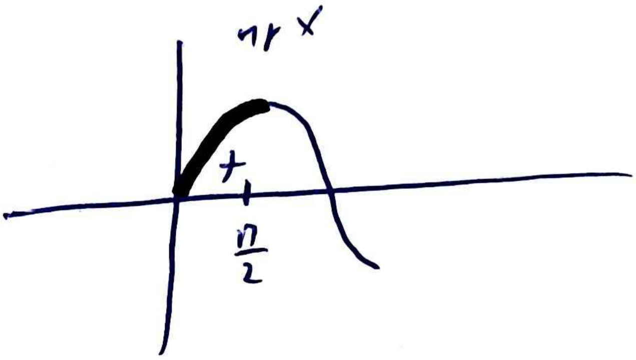
$f'(\alpha) < f'(\xi) < f'(\beta)$

$$e^{\alpha} < \frac{e^{\beta}-e^{\alpha}}{\beta-\alpha} < e^{\beta}.$$

Ⓑ νδο $(\beta-\alpha)\text{συν}\beta < \text{ημ}\beta-\text{ημ}\alpha < (\beta-\alpha)\text{συν}\alpha$;

$$\text{συν}\beta < \frac{\text{ημ}\beta-\text{ημ}\alpha}{\beta-\alpha} < \text{συν}\alpha \,..$$

$$\boxed{f(x)=\text{ημ}x}$$



$f'(x)=\text{συν}x$

$f''(x)=-\text{ημ}x<0$

$f'\downarrow$

$$f'(\xi)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=\frac{\text{ημ}\beta-\text{ημ}\alpha}{\beta-\alpha}.$$

$\alpha<\xi<\beta$

$f'\downarrow$

$f'(\alpha)>f'(\xi)>f'(\beta)$

$$\text{συν}\alpha > \frac{\text{ημ}\beta-\text{ημ}\alpha}{\beta-\alpha} > \text{συν}\beta .$$

19. $f''(x) \neq 0 \quad \forall x \in R.$

f 2 φορές παρ/μη.

f'' συνεχής.

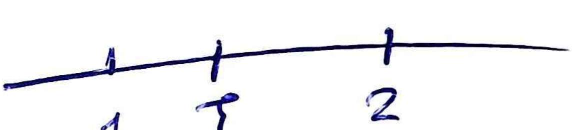
$f''(x) > 0$ ή $f''(x) < 0$ $\forall x \in R.$

$f(2) - f(1) > f'(1)$

ΘΜΤ για την f στο (1,2)

$$f'(\xi) = \frac{f(2)-f(1)}{2-1} = f(2)-f(1)$$

$f'(\xi) > f'(1)$



$f'(\xi) - f'(1) > 0$

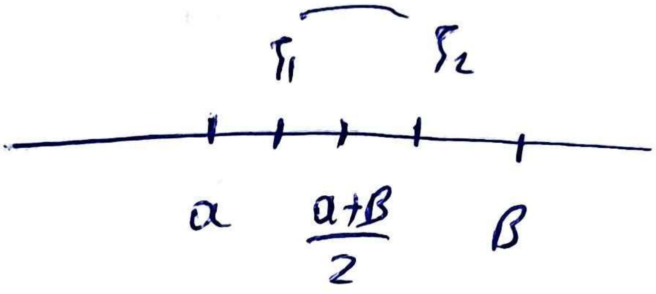
ΘΜΤ για την f' στο (1,ξ).

$$f''(\rho) = \frac{\overset{+}{f'(\xi) - f'(1)}}{\underset{+}{\xi - 1}} > 0 \qquad f''(x) > 0$$

28. ⓐ $f:[a,\beta] \longrightarrow R$ παρ/μη.

$f' \nearrow$

νδο $f\left(\frac{a+\beta}{2}\right) < \frac{f(a)+f(\beta)}{2}$.



$$f'(\xi_1) = \frac{f\left(\frac{a+\beta}{2}\right) - f(a)}{\frac{a+\beta}{2} - a} = \frac{f\left(\frac{a+\beta}{2}\right) - f(a)}{\frac{\beta - a}{2}} = 2\,\frac{f\left(\frac{a+\beta}{2}\right) - f(a)}{\beta - a}$$

$$f'(\xi_2) = \frac{f(\beta) - f\left(\frac{a+\beta}{2}\right)}{\beta - \frac{a+\beta}{2}} = 2\,\frac{f(\beta) - f\left(\frac{a+\beta}{2}\right)}{\beta - a}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$\cancel{2}\,\frac{f\left(\frac{a+\beta}{2}\right) - f(a)}{\cancel{\beta - a}} < \cancel{2}\,\frac{f(\beta) - f\left(\frac{a+\beta}{2}\right)}{\cancel{\beta - a}}$$

$$2f\left(\frac{a+\beta}{2}\right) < f(\beta) + f(a)$$ ✓

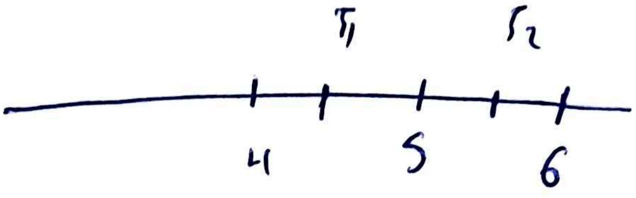
28.

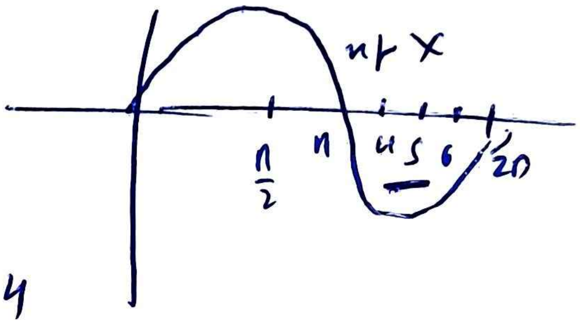
Ⓑ νδο $ημ4 + ημ6 > 2ημ5$

$f(x) = ημx$

$f'(x) = συνx \quad f' \nearrow$

$f''(x) = -\underset{\ominus}{ημx} > 0$





$$f'(ξ_1) = \frac{f(5) - f(4)}{5-4} = ημ5 - ημ4$$

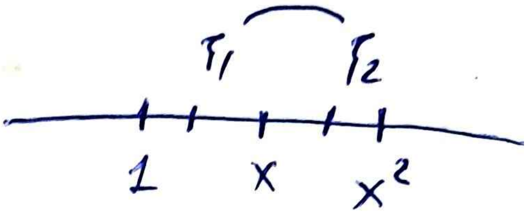
$$f'(ξ_2) = \frac{f(6) - f(5)}{6-5} = ημ6 - ημ5$$

$$ξ_1 < ξ_2 \Rightarrow f'(ξ_1) < f'(ξ_2)$$

$$ημ5 - ημ4 < ημ6 - ημ5$$

$$2ημ5 < ημ6 + ημ4$$

22. ⓐ $(x+1)\,f(x) < x\,f(1) + f(x^2)$ $\qquad \underline{\underline{x>1}}$



$$f'(\xi_1) = \frac{f(x)-f(1)}{x-1}$$

$$f'(\xi_2) = \frac{f(x^2)-f(x)}{x^2-x}$$

$$\xi_1 < \xi_2 \quad \xRightarrow{f'\nearrow} \quad f'(\xi_1) < f'(\xi_2)$$

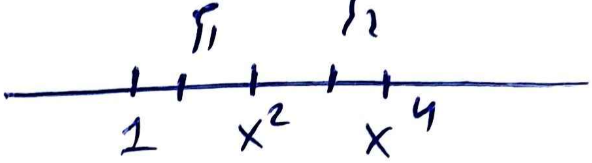
$$\frac{f(x)-f(1)}{\cancel{x-1}} < \frac{f(x^2)-f(x)}{x\cancel{(x-1)}}$$

$$x\,f(x) - x\,f(1) < f(x^2) - f(x).$$

$$\boxed{x\,f(x) + f(x) < f(x^2) + x\,f(1)}$$

$$\underline{\underline{(x+1)\,f(x) < f(x^2) + x\,f(1)}}$$

Ⓑ νδο $(x^2+1)\,f(x^2) < x^2 f(1) + f(x^4)$ $\forall x>1$.



$$f'(\xi_1) = \frac{f(x^2)-f(1)}{x^2-1}$$

$$f'(\xi_2) = \frac{f(x^4)-f(x^2)}{x^4-x^2}$$

$$\xi_1 < \xi_2 \xRightarrow{f'\uparrow} f'(\xi_1) < f'(\xi_2)$$

$$\frac{f(x^2)-f(1)}{\cancel{x^2-1}} < \frac{f(x^4)-f(x^2)}{x^2\cancel{(x^2-1)}}$$

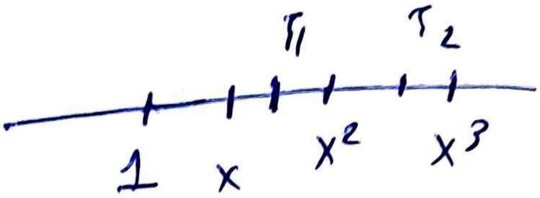
• $x>1 \Rightarrow x^2>1 \Rightarrow x^2-1>0$

$$x^2 f(x^2) - x^2 f(1) < f(x^4) - f(x^2).$$

$$x^2 f(x^2) + f(x^2) < f(x^4) + x^2 f(1).$$

$$\boxed{(x^2+1)\,f(x^2) < f(x^4) + x^2 f(1)}$$

22. (γ) $x f(x) + f(x^3) > (x+1) f(x^2) \quad \forall x > 1.$



$$f'(\xi_1) = \frac{f(x^2) - f(x)}{x^2 - x}$$

$$f'(\xi_2) = \frac{f(x^3) - f(x^2)}{x^3 - x^2}$$

$$\xi_1 < \xi_2 \Rightarrow \frac{f(x^2) - f(x)}{x^2 - x} < \frac{f(x^3) - f(x^2)}{x^3 - x^2}$$

$$\frac{f(x^2) - f(x)}{\cancel{x(x-1)}} < \frac{f(x^3) - f(x^2)}{x^{\cancel{2}}\cancel{(x-1)}}$$

$$x f(x^2) - x f(x) < f(x^3) - f(x^2)$$

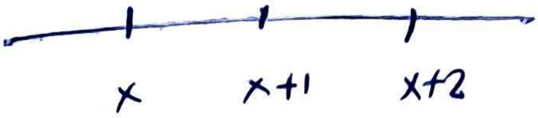
$$x f(x^2) + f(x^2) < f(x^3) + x f(x)$$

$$f(x^2)(x+1) < f(x^3) + x f(x)$$

29. $f:\mathbb{R}\to\mathbb{R}$ δύο φορές παρ/μη.

(β1) $f(x) + f(x+2) = 2f(x+1)$.

$$f(x+2) - f(x+1) = f(x+1) - f(x)$$



$$\frac{f(x+2) - f(x+1)}{(x+2) - (x+1)} = \frac{f(x+1) - f(x)}{x+1 - x}$$

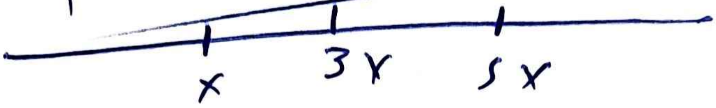
$$\underbrace{f'(\xi_2) = f'(\xi_1).}_{\text{Rolle}}$$

$$f''(\xi) = 0 .$$

(B). $f(x) + f(5x) = 2f(3x)$

$f(x) + f(5x) = f(3x) + f(3x)$

$$\boxed{f(5x) - f(3x) = f(3x) - f(x)}$$



$$f'(\xi_1) = \frac{f(3x) - f(x)}{3x - x} = \frac{f(3x) - f(5x)}{2x} =$$

$$f'(\xi_2) = \frac{f(5x) - f(3x)}{5x - 3x} = \frac{f(5x) - f(3x)}{2x}.$$

$$f'(\xi_1) = f'(\xi_2)$$

Rolle

$\exists\, \xi \in (\xi_1, \xi_2)$

T.L $f''(\xi) = 0$

·31

16. $f: \mathbb{R} \to \mathbb{R}$ συνεχής.

$f(0) = 1$

$\boxed{2 \le f'(x) \le 4 \quad \forall x \in \mathbb{R}}$

Ν.δ.ο. $11 \le f(5) \le 21$.

$$f'(\xi) = \frac{f(5) - f(0)}{5 - 0}$$

$$2 \le f'(\xi) \le 4$$

$$2 \le \frac{f(5) - f(0)}{5} \le 4$$

$$10 \le f(5) - f(0) \le 20$$

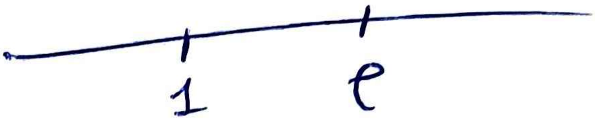
$$10 \le f(5) - 1 \le 20$$

$$11 \le f(5) \le 21$$

18. $f(e) = e\ln 3$

$f'(x) < \ln 3$

νδο $f(1) > \ln 3$



$$f'(\xi) = \frac{f(e) - f(1)}{e-1}$$

$$f'(\xi) < \ln 3$$

$$\frac{f(e) - f(1)}{e-1} < \ln 3$$

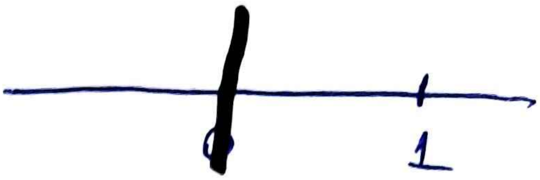
$$\frac{e\ln 3 - f(2)}{e-1} < \ln 3$$

$$e\ln 3 - f(1) < \ln 3(e-1)$$

$$\cancel{e\ln 3} - f(1) < \cancel{\ln 3 e} - \ln 3$$

$$f(1) > \ln 3$$

14. $f(x) = \begin{cases} x^3+\alpha, & x<0 \\ 0, & x=0 \\ x^2, & x>0, \end{cases}$



ⓐ Η f συνεχής στο $[0,1]$

$\lim\limits_{x\to 0^+} f(x) = 0$ Άρα συνεχής στο $[0,1]$

$f(0) = 0$

Η f παρ/μη στο $(0,1)$ ως π.π.σ.

ⓑ · Πρέπει και αρκεί συνεχής στο 0.

$$\left.\begin{aligned} &\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^3+a = a. \\ &\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0 \end{aligned}\right\} \underline{\underline{a=0}}$$

ⓖ $y - f(\xi) = f'(\xi)(x-\xi)$ // ΕΑΒ

$f'(\xi) = \lambda_{AB} = \dfrac{f(+1)-f(-1)}{1-(-1)}$ που

ισχύει από ΘΜΤ στο $[-1,1]$.

$$f'(x) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{1+1}{2} = 1 .$$

$$f'(x) = 1$$

$\underline{x<0}$

$f'(x) = 1 .$

$3x^2 = 1 .$

$x^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$

$$\boxed{x = -\frac{\sqrt{3}}{3}}$$

$\underline{x>0}$

$f'(x) = 1 .$

$2x = 1$

$$\boxed{x = \frac{1}{2}}$$

23. Αν $|f'(x)| \geq 1$ $\forall x \in [1,3]$.

νδο $(AB) \geq 2\sqrt{2}$.

$$(AB) = \sqrt{(3-1)^2 + (f(3)-f(1))^2} = \sqrt{4+(f(3)-f(1))^2}$$

νδο $\sqrt{4+(f(3)-f(1))^2} \geq 2\sqrt{2}$

$$4+(f(3)-f(1))^2 \geq 8$$

$$(f(3)-f(1))^2 \geq 4$$

$$(f(3)-f(1))^2 \geq 2^2$$

$$|f(3)-f(1)| \geq |2|$$

Αρκει νδο $|f(3)-f(1)| \geq 2$

$$f'(\xi) = \frac{f(3)-f(1)}{3-1} = \frac{f(3)-f(1)}{2}$$

ομως $|f'(x)| \geq 1 \Rightarrow |f'(\xi)| \geq 1 \Rightarrow \left|\frac{f(3)-f(1)}{2}\right| \geq 1$

$|f(3)-f(1)| \geq 2$ ✓

25. (α) Νδο $|ημ\sqrt{x^2+1} - ημx| \leq |\sqrt{x^2+1} - x|$

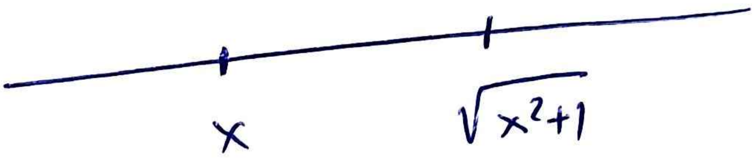
- $\sqrt{x^2+1} > x$

1η: Αν $x \geq 0$ τότε $\sqrt{x^2+1}^2 \geq x^2$

$\cancel{x^2}+1 \geq \cancel{x^2}$

$1 \geq 0$

2η: Αν $x < 0$ τότε ισχύει!.



$f(x) = ημx$

$$f'(\xi) = \frac{f(\sqrt{x^2+1}) - f(x)}{\sqrt{x^2+1} - x} = \frac{ημ\sqrt{x^2+1} - ημx}{\sqrt{x^2+1} - x}.$$

$$\left| \frac{ημ\sqrt{x^2+1} - ημx}{\sqrt{x^2+1} - x} \right| \leq 1$$

Αρκει νδο $|f'(ξ)| \leq 1$.

$$f(x) = ημx$$

$$f'(x) = συνx$$

$$-1 \leq συνx \leq 1$$

$$-1 \leq f'(x) \leq 1$$

$$|f'(x)| \leq 1.$$

$|f'(ξ)| \leq 1$. οεδ

27. $f:(0,+\infty)\longrightarrow R$

$f(1)=1$

$f^2(x)=x-\ln x \quad \forall x>0.$

$f^2(x)=\sqrt{x-\ln x}^{\,2}$

$\ln x\le x-1$

$1\le x-\ln x$

$0<x-\ln x$

$|f(x)|=|\overset{+}{\sqrt{x-\ln x}}|$

$|\overset{+}{f(x)}|=\sqrt{x-\ln x}$

$f(x)=\sqrt{x-\ln x}$

P,W $f(x)$

$f(x)=0$

$|f(x)|=0$

$\sqrt{x-\ln x}=0$

A contr.

$f(x)>0$ și $f(x)<0 \quad \forall x>0.$

$f(1)=1$

$f(x)>0$

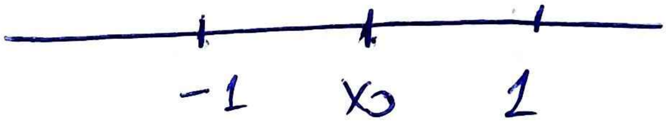
32. $\underline{\underline{f(1)=1}}$ $\underline{\underline{f(-1)=-1}}$

ⓐ νδο $\exists x_0 \in (-1,1)$ τ.ω $f(x_0)=0$

$$\left.\begin{array}{l} f(-1)=-1 \\ f(1)=1 \end{array}\right\} \quad f(-1)\cdot f(1)<0$$

Bolzano $\exists x_0 \in (-1,1)$ τ.ω $f(x_0)=0$.

Ⓑ νδο $\exists \xi_1, \xi_2 \in (-1,1)$ τ.ω $\dfrac{1}{f'(\xi_1)} + \dfrac{1}{f'(\xi_1)} = 2$.



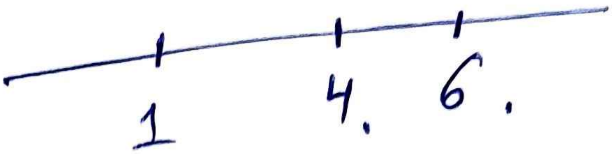
$$f'(\xi_1) = \frac{f(x_0)-f(-1)}{x_0+1} = \frac{1}{x_0+1}$$

$$f'(\xi_2) = \frac{f(1)-f(x_0)}{1-x_0} = \frac{1-0}{1-x_0} = \frac{1}{1-x_0}$$

$$\frac{1}{f'(\xi_1)} + \frac{1}{f'(\xi_2)} = \frac{1}{\frac{1}{x_0+1}} + \frac{1}{\frac{1}{1-x_0}} =$$

$$= \frac{x_0+1}{1} + \frac{1-x_0}{1} = 2.$$

34. $f(1)=f(6)$



νδο $3f'(\xi_1)+2f'(\xi_2)=0$.

$\Delta = 6-1 = 5.$

$\delta_1 = \frac{k_1}{k}\cdot\Delta = \frac{3}{\cancel{5}}\cdot\cancel{5}$

$k_1=3$, $k_2=2$ $\Big\}$ $k=3+2=5$.

$\delta_1 = 3.$

$$f'(\xi_1)=\frac{f(4)-f(1)}{4-1}=\frac{f(4)-f(1)}{3}$$

$$f'(\xi_2)=\frac{f(6)-f(4)}{6-4}=\frac{f(6)-f(4)}{2}$$

$$3f'(\xi_1)+2f'(\xi_2)=\cancel{3}\frac{\cancel{f(4)}-f(1)}{\cancel{3}}+\cancel{2}\frac{f(6)-\cancel{f(4)}}{\cancel{2}}$$

$$=f(6)-f(1)=0$$ ✓

Συνέπειες ΘΜΤ

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1. Αν f συνεχής στο Δ

Αν $f'(x)=0$ $\forall x$ εσωτερικό του Δ.

Τότε $f(x)=c$ σταθερή συνάρτηση

2. Αν f,g συνεχείς στο Δ.

Αν $f'(x)=g'(x)$ $\forall x$ εσωτερικό του Δ.

Τότε $f(x)=g(x)+c$.

3. Αν $f'(x)=f(x)$ $\forall x \in R$.

Τότε $f(x)=ce^{x}$.

4. $f'(x) = f(x) + 2xe^x$, $\forall x \in \mathbb{R}$.

ⓐ. Νδο $g(x) = \dfrac{f(x)}{e^x} - x^2$ σταθερη

$$g'(x) = \frac{f'(x)e^x - f(x)e^x}{e^{2x}} - 2x$$

$$g'(x) = \frac{f'(x) - f(x)}{e^x} - 2x$$

$$g'(x) = \frac{f'(x) - f(x) - 2xe^x}{e^x}$$

$$g'(x) = \frac{\cancel{f(x)} + 2x\cancel{e^x} - \cancel{f(x)} - 2\cancel{xe^x}}{e^x} = 0$$

$$g'(x) = 0 \Rightarrow g(x) = C$$

(B) Αν $f(0)=1$ βρεί τύπο $f(x)$

$$g(x) = c$$

$$\frac{f(x)}{e^x} - x^2 = c$$

$x=0$

$$\frac{f(0)}{1} - 0 = c$$

$$f(0) = c$$

$$1 = c$$

$$\frac{f(x)}{e^x} - x^2 = 1 \Leftrightarrow \boxed{f(x) = (x^2+1)e^x}$$

Επομενο Μαθημα

Σελ 34

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⑳

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㉛ .

㉝ -

11. [C] 34

Ⓑ $f(x) = \ln(x - \ln x)$

Πρπ $x > 0$ και $x - \ln x > 0$

• $\ln x \leq x - 1$

$1 \leq x - \ln x$

$0 < x - \ln x$

$D_f = (0, +\infty)$.

Ⓖ $f(x) = \dfrac{x}{e^x - x - 1}$ $D_f = \mathbb{R}^*$.

Πρπ $e^x - x - 1 \neq 0$

• $e^x \geqslant x + 1$

$e^x - x - 1 \geqslant 0$

("=" αν $x = 0$)

12. (α) Νδο $e^{\frac{4}{5}} > \frac{25}{16}$

• $e^{x} \geqslant x+1$

$e^{\frac{4}{5}} \geqslant \frac{4}{5}+1$

$e^{\frac{4}{5}} \geqslant \frac{4}{5}+\frac{5}{5} \iff e^{\frac{4}{5}} \geqslant \frac{9}{5}$

Αρκει νδο $\frac{9}{5} > \frac{25}{16}$

$9 \cdot 16 > 5 \cdot 25$

$144 > 125$ ✓

Αρα $e^{\frac{4}{5}} > \frac{25}{16}$.

(β) νδο $\ln 2 < \frac{2}{e}$

Αρκει νδο

$\ln x \leq x-1$

$\ln 2 \leq 2-1$

$\ln 2 \leq 1 < \frac{2}{e}$ $\Rightarrow \ln 2 < \frac{2}{e}$

∘ $1 > \frac{2}{e}$

12. Ⓘ $e^{x^2} - x^2 > 0$

• $e^x \geq x + 1$

$e^{x^2} \geq x^2 + 1$

$\underline{\underline{e^{x^2} > x^2}}$ $\Rightarrow e^{x^2} - x^2 > 0$

(η) $\ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$, $x > 0$.

$\ln x \leq x - 1$ Το "=" για $x = 1$

$\ln\left(1 + \frac{1}{x}\right) < 1 + \frac{1}{x} - 1$ $\Rightarrow \cancel{1} + \frac{1}{x} = \cancel{1}$

$\frac{1}{x} = 0$

Αδυνατη

$\ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}$.

13. Ⓑ $e^{1-x} = \frac{1}{x}$

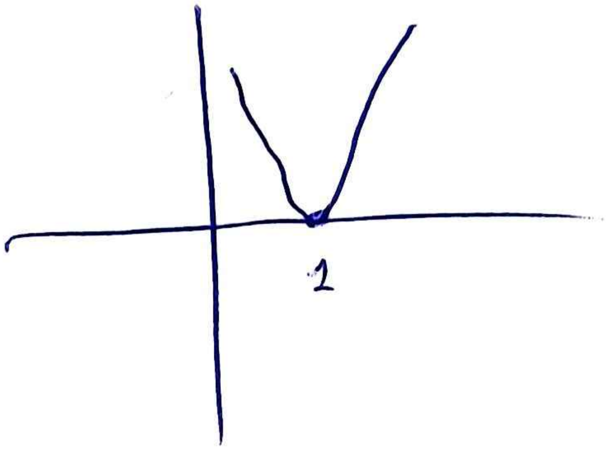
$$e^{1-x} - \frac{1}{x} = 0$$

$$\frac{e}{e^x} - \frac{1}{x} = 0$$

$$\frac{e}{e^x} = \frac{1}{x}$$

$$ex = e^x$$

$$0 = e^x - ex$$



$$\boxed{f(x) = e^x - ex}$$

$$f'(x) = e^x - e$$

$\rightarrow e^x - e = 0$

$$e^x = e$$

$$\boxed{x=1}$$

x		1	
f'	$-$	0	$+$
f	↘		↗

$$f(x) \geq f(1)$$

$$f(x) \geq 0$$

$$e^x - ex \geq 0$$

$$\boxed{x=1.}$$

15. $f: [1,3] \to \mathbb{R}$ συνεχ & $\begin{cases} f(1)=2 \\ f(3)=0 \end{cases}$

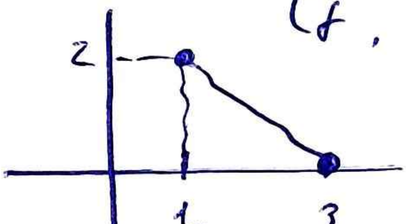
$\Sigma T_f = [0,2]$

νδο $\exists \xi$ ώστε $y - f(\xi) = f'(\xi)(x-\xi) \perp y=x-1$

$f'(\xi) \cdot 1 = -1$

$$\boxed{f'(\xi) = -1}$$

Νδο $\exists \xi \in (1,3)$ τ.ω $f'(\xi) = -1$



Bolzano

$\underbrace{f'(x)+1}_{g(x)} = 0$

$g(1) = f'(1)+1$

$g(3) = f'(3)+1$

Rolle

$g(x) = f'(x)+1$

$G(x) = f(x)+x$

$\left.\begin{aligned} G(1) &= f(1)+1 = 2+1 = 3 \\ G(3) &= f(3)+3 = 0+3 = 3 \end{aligned}\right\}$ Rolle

$\exists \xi \in (1,3)$

τ.ω $G'(\xi) = 0$

$g(\xi) = 0$

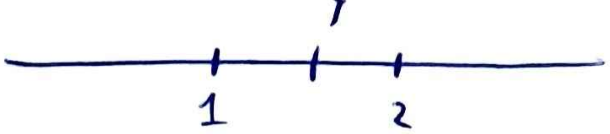
$f'(\xi)+1 = 0 \to f'(\xi) = -1$

ΘΜΤ

$$f'(\xi) = \frac{f(3) - f(1)}{3 - 1} = \frac{0 - 2}{2} = -1$$ ✓

17. $f(1) = e \ln 2$

$f'(x) < \ln 2$ Νδο $f(2) < \ln 2^{e+1}$



$$f'(\xi) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1)$$

$f'(\xi) < \ln 2 \Rightarrow f(2) - f(1) < \ln 2$

$f(2) - e\ln 2 < \ln 2$

$f(2) < \ln 2 + e \ln 2$

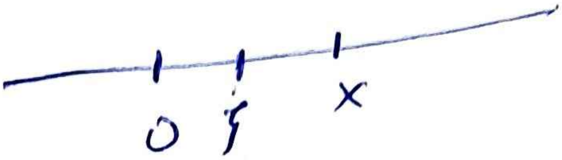
$f(2) < \ln 2 (e+1)$

$f(2) < (e+1) \ln 2$

$f(2) < \ln 2^{e+1}$

20. $f' \nearrow$ $f(0)=0$ $f'(x)>1$

Nδo $x < f(x) < x f'(x)$ $\forall x > 0$?



$$f'(\xi) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x) \Rightarrow \frac{f(x)}{x} < f'(x)$$

$$\boxed{f(x) < x f'(x)}$$

$$f'(\xi) > 1$$

$$\frac{f(x)}{x} > 1$$

$$x < f(x) < x f'(x)$$

$$\boxed{f(x) > x}$$

21. Η C_f εφάπτεται στο x'x $f'\nearrow$

στο $(0,0)$. $\searrow$ $y=0$

$y = 0 \cdot x + 0$

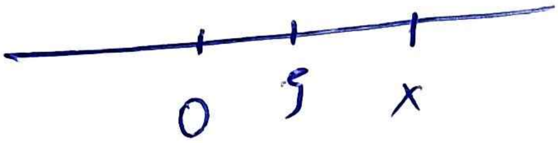
$$\begin{cases} f'(0)=0 \\ f(0)=0 \end{cases}$$

$y = \alpha x + \beta$

$M(x_0, f(x_0))$

$$\begin{cases} f'(x_0)=\alpha \\ f(x_0)=\alpha x_0+\beta \end{cases}$$

(α) νδο $x f'(x) > f(x)$, $x>0$



$$f'(\xi) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x) \Rightarrow \frac{f(x)}{x} < f'(x)$$

$$f(x) < x f'(x)$$

(β) νδο η $\frac{f(\beta)-\beta f'(\beta)}{x-2} - \frac{f'(\beta)}{x-1} = 0$ $\beta>0$ $(1,2)$

$$\boxed{g(x) = (x-1)\left[f(\beta)-\beta f'(\beta)\right] - (x-2) f'(\beta)}$$

$g(1) = f'(\beta) > 0$

$g(2) = f(\beta) - \beta f'(\beta) < 0$

$$\boxed{\begin{array}{l} f(x) < x f'(x) \quad \forall x > 0 \\ f(\beta) < \beta f'(\beta) \quad \beta > 0 \\ f(\beta) - \beta f'(\beta) < 0 \\ \qquad g(2) < 0 \end{array}}$$

- $\beta > 0 \xrightarrow{f' \uparrow} f'(\beta) > f'(0) \Rightarrow f'(\beta) > 0$

$g(1)\,g(2) < 0$

Bolzano $\exists\, \xi \in (1,2)$ τω

$g(\xi) = 0$

24. $|f'(x)| \leq 2 \quad \forall x \in \mathbb{R}$.

Ⓑ $|f(\beta) - f(\alpha)| \leq 2|\beta - \alpha| \quad \forall \alpha, \beta, \alpha < \beta$

$$f'(\xi) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$$

$$|f'(\xi)| \leq 2 \Rightarrow \left| \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \right| \leq 2$$

$$\frac{|f(\beta) - f(\alpha)|}{|\beta - \alpha|} \leq 2 \Rightarrow |f(\beta) - f(\alpha)| \leq 2|\beta - \alpha|.$$

Ⓑ $\lim\limits_{x \to +\infty} f(\sqrt{x+1} - f(\sqrt{x}))$.

- $\sqrt{x} < \sqrt{x+1}$

Άρα $|f(\sqrt{x+1}) - f(\sqrt{x})| \leq 2|\sqrt{x+1} - \sqrt{x}|$

$$-2(\sqrt{x+1} - \sqrt{x}) \leq f(\sqrt{x+1}) - f(\sqrt{x}) \leq 2(\sqrt{x+1} - \sqrt{x})$$

• $\lim\limits_{x \to +\infty} -2\left(\sqrt{x+1}-\sqrt{x}\right) = \lim\limits_{x \to +\infty} -2 \dfrac{1}{\sqrt{x+1}+\sqrt{x}} = 0$

• $\lim\limits_{x \to +\infty} 2\left(\sqrt{x+1}-\sqrt{x}\right) = \lim\limits_{x \to +\infty} 2 \dfrac{1}{\sqrt{x+1}+\sqrt{x}} = 0$

Άρα K, Λ

$$\lim_{x \to +\infty} f(\sqrt{x+1}) - f(\sqrt{x}) = 0$$

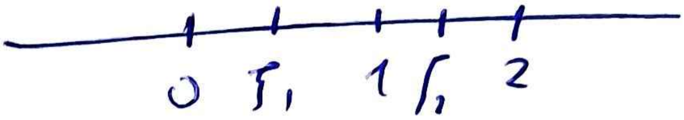
26. $f: [0,2] \to \mathbb{R}$ συνεχής.

$f(0) = -1$, $f(2) = 1$ $\Big\}$ $f(0)f(2) < 0$

Bolzano $\exists \xi \in (0,2)$ τ.ω $f(\xi) = 0$

$f'(x) \geq 1$

νδο $f(1) = 0$



$$f'(\xi_1) = \frac{f(1)-f(0)}{1-0} = f(1)+1$$

$$f'(\xi_2) = \frac{f(2)-f(1)}{2-1} = 1-f(1)$$

- $f'(\xi_1) \geq 1 \Rightarrow f(1)+1 \geq 1 \Rightarrow f(1) \geq 0$
- $f'(\xi_2) \geq 1 \Rightarrow 1-f(1) \geq 1 \Rightarrow -f(1) \geq 0$

$f(1) \leq 0$

$$\boxed{f(1) = 0}$$

90.

$$f'(\xi_1)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha}$$

$$f'(\xi_2)=\frac{f(\gamma)-f(\beta)}{\gamma-\beta}$$

Αφου ειναι συνευθειακα τα Α,Β,Γ.

ε_{AB}, $\varepsilon_{B\Gamma}$ ταυτιζονται

$\Leftrightarrow \lambda_{AB}=\lambda_{B\Gamma}$

$f'(\xi_1)=f'(\xi_2)$

Rolle

$\exists\, \xi\in(\xi_1,\xi_2)$

τω $f''(\xi)=0$.

31. $f(1)=2$ $\quad f(2)=4$

ⓐ νδο $\exists x_0 \in (1,2)$ τ.ω $f(x_0)=2(3-x_0)$

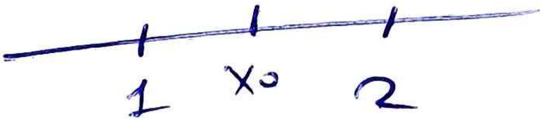
$$f(x)=2(3-x)$$

$$\underbrace{f(x)-2(3-x)}_{g(x)}=0$$

$$\left.\begin{array}{l} g(1)=f(1)-4=2-4=-2 \\ g(2)=f(2)-2=4-2=2 \end{array}\right\} g(1)g(2)<0$$

Bolzano $\exists \xi \in (1,2)$ τ.ω $g(\xi)=0$.

$f(x_0)=2(3-x_0)$.



Ⓑ $f'(\xi_1)=\dfrac{f(x_0)-f(1)}{x_0-1}$

$$f'(\xi_2)=\frac{f(2)-f(x_0)}{2-x_0}$$

$$f'(\xi_1)f'(\xi_2)=\frac{f(x_0)-f(1)}{x_0-1}\cdot\frac{f(2)-f(x_0)}{2-x_0}=\frac{2(3-x_0)-2}{x_0-1}\cdot\frac{4-f(x_0)}{2-x_0}$$

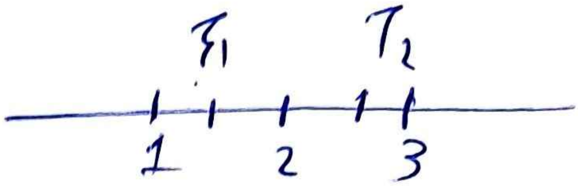
$$= \frac{4 - 2x_0}{x_0 - 1} \cdot \frac{4 - 2(3 - x_0)}{2 - x_0} =$$

$$= \frac{2(\cancel{2 - x_0})}{x_0 - 1} \cdot \frac{2x_0 - 2}{\cancel{2 - x_0}} = \frac{2}{\cancel{x_0 - 1}} \cdot \frac{2(\cancel{x_0 - 1})}{1}$$

$$= 4 .$$

33. $f \downarrow$

$f'(\xi_1) + f'(\xi_2) = -2$



$$f'(\xi_1) = \frac{f(2)-f(1)}{2-1} = f(2)-f(1)$$

$$f'(\xi_2) = \frac{f(3)-f(2)}{3-2} = f(3)-f(2)$$

$$f'(\xi_1) + f'(\xi_2) = \cancel{f(2)} - f(1) + f(3) - \cancel{f(2)}$$

$$= f(3) - f(1) = 0 - 2 = -2$$

✓

$$\left.\begin{array}{l} D_f = [1,3] \\ f \downarrow \\ \{T_f\ [0,2] \end{array}\right\} \begin{array}{l} f(1) = 2 \\ f(3) = 0 \end{array}$$

7. $f: (0,+\infty) \to \mathbb{R}$. παρ/μη.

$f(1)=1$

$x^2 f'(x) + x f(x) = 1$.

Νδο $f(x) = \frac{1+\ln x}{x}$.

α' τροπος

$$f(x) = \frac{1+\ln x}{x} \iff x f(x) = 1 + \ln x$$

$$\underbrace{x f(x) - 1 - \ln x}_{g(x)} = 0$$

Αρκει νδο $g(x)=0$

$$g'(x) = f(x) + x f'(x) - \frac{1}{x} = \frac{x f(x) + x^2 f'(x) - 1}{x}$$

$$= \frac{1-1}{x} = 0$$

$\to g(x) = c$

$g(1) = c$

$f(1) - 1 - 0 = c$

$1 - 1 = c$

$c = 0 \quad \to g(x) = 0 \Rightarrow f(x) = \frac{1+\ln x}{x}$.

Β' τροπος

$x > 0$

$$x^2 f'(x) + x f(x) = 1$$

$$\underbrace{x f'(x) + f(x)} = \frac{1}{x}$$

$$(x f(x))' = (\ln x)'$$

$$x f(x) = \ln x + c$$

$x = 1$

$$f(1) = \ln 1 + c$$

$$1 = c$$

$$x f(x) = \ln x + 1$$

$$\boxed{f(x) = \frac{\ln x + 1}{x}}$$

12. $f'(x) = 3f(x) + 2xe^{3x}$ $\forall x \in \mathbb{R}$.

Nδο $f(x) = x^2e^{3x}$.

$f(0)=0$

$$\boxed{g(x) = f(x) - x^2e^{3x}}$$

$$g'(x) = f'(x) - \left(2xe^{3x} + x^2 3e^{3x}\right)$$

$$g'(x) = f'(x) - 2xe^{3x} - 3x^2e^{3x}$$

$$g'(x) = 3f(x) + \cancel{2xe^{3x}} - \cancel{2xe^{3x}} - 3x^2e^{3x}$$

$$g'(x) = 3f(x) - 3x^2e^{3x}.$$

$$g'(x) = 3\left(f(x) - x^2e^{3x}\right)$$

$$g'(x) = 3g(x)$$

Δω Βγαινει ◯ Εχει προβλημα.

$$f'(x) - 3f(x) = 2xe^{3x}$$

- $g(x) = -3$
- $G(x) = -3x$
- $e^{G(x)} = e^{-3x}$

$$\downarrow$$

$$\underbrace{e^{-3x} f'(x) - 3e^{-3x} f(x)} = 2xe^{-3x}e^{3x}$$

$$\left(e^{-3x} f(x)\right)' = 2x$$

$$\left(e^{-3x} f(x)\right)' = \left(x^2\right)'$$

$$e^{-3x} f(x) = x^2 + C$$

$$e^0 f(0) = 0 + C$$

$$0 = C$$

$$e^{-3x} f(x) = x^2$$

$$\frac{f(x)}{e^{3x}} = x^2$$

$$\underline{\underline{f(x) = x^2 e^{3x}}}$$

Γραμμικη διαφορικη εξισωση

$$f'(x) + g(x) f(x) = h(x)$$ (Μορφη)

- $g(x)$
- Βρισκω παραγουσα $G(x)$
- Πολ/ζω ποωτον με $e^{G(x)}$

$$e^{G(x)} f'(x) + g(x) e^{G(x)} f(x) = h(x) e^{G(x)}$$

$$\left(e^{G(x)} f(x)\right)' = h(x) e^{G(x)}$$

9. $f(0)=0$. $\quad x f(x) + x^2 f'(x) + f'(x) = \sqrt{x^2+1}$

$$x\big(f(x) + x f'(x)\big) = \sqrt{x^2+1} - f'(x).$$

νδο $\quad f(x) = \dfrac{x}{\sqrt{x^2+1}}$

$$\underbrace{\sqrt{x^2+1}\, f(x) - x}_{g(x)} = 0$$

Αρκεί νδο

$g(x) = 0$

$$g'(x) = \frac{2x}{2\sqrt{x^2+1}} f(x) + \sqrt{x^2+1}\, f'(x) - 1$$

$$g'(x) = \frac{x f(x) + (x^2+1) f'(x) - \sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$g'(x) = \frac{x f(x) + x^2 f'(x) + f'(x) - \sqrt{x^2+1}}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1} - \sqrt{x^2+1}}{\sqrt{x^2+1}} = 0$$

$\Rightarrow g'(x) = 0 \quad \Rightarrow g(x) = C.$

$$\sqrt{x^2+1}\, f(x) - x = c$$

$$\underline{x = 0}$$

$$f(0) - 0 = c$$

$$c = 0.$$

$$\sqrt{x^2+1}\, f(x) - x = 0$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

Επομαο Μαθημα

Σελ 34

⑪ α δ

⑫ α γ ε στ

⑬ α

Σελ 47

②

③

⑤

⑥

⑧

⑩

⑪