Θεμα 1

- $f(x) = \ln x + 1 \qquad D_f = (0, +\infty)$
- $g(x) = \dfrac{\alpha - 1}{x}$
- $g(1) = 1 \iff \dfrac{\alpha - 1}{1} = 1 \iff \alpha - 1 = 1 \iff \alpha = 2$ αρα $g(x) = \dfrac{1}{x}$, $D_g = \mathbb{R}^*$

(α) Βρες τα x ωστε η C_g να ειναι πανω

απο την $\varepsilon: y = 1$

$$g(x) > 1 \iff \frac{1}{x} > 1 \iff \frac{1}{x} - 1 > 0 \iff \frac{1-x}{x} > 0$$

x		0		1	
$1-x$	$+$	‖	$+$	0	$-$
x	$-$	‖	$+$		$+$
$\frac{1-x}{x}$	$-$	‖	$+$		$-$

Οταν $x \in (0, 1)$ η C_g πανω ε.

(β) Σχετικη θεση

$$f(x) = g(x) \iff \ln x + 1 = \frac{1}{x} \iff \underbrace{\ln x + 1 - \frac{1}{x}}_{L(x)} = 0$$

$$L(x) = 0 \iff L(x) = L(1) \overset{L \text{ 1-1}}{\iff} \boxed{x = 1}$$

Μονοτονια

$$L'(x) = \frac{1}{x} + \frac{1}{x^2} > 0$$

$L \nearrow \Rightarrow L$ 1-1

Όταν $x<1 \Rightarrow L(x)<L(1) \Leftrightarrow f(x)-g(x)<0$
$\Rightarrow f(x)<g(x)$

Όταν $x>1 \Rightarrow f(x)>g(x)$.

(γ) $h = g\circ f$.

$$h(x)=(g\circ f)(x)=g(f(x))=\frac{1}{\ln x+1}$$

$x\in D_f$ και $f(x)\in D_g$ $\qquad D_h=(0,\frac{1}{e})\cup(\frac{1}{e},+\infty)$

$x>0$ $\qquad \ln x+1\neq 0$

$\ln x\neq -1$

$x\neq e^{-1} \Rightarrow x\neq \frac{1}{e}$

(δ) $h(x_1)=h(x_2) \Leftrightarrow \frac{1}{\ln x_1+1}=\frac{1}{\ln x_2+1}$

$\Leftrightarrow \ln x_1+1=\ln x_2+1 \Leftrightarrow \ln x_1=\ln x_2 \Leftrightarrow x_1=x_2$

h είναι 1-1 άρα αντιστρέψιμη.

Εύρεση αντίστροφης

$y=h(x) \Leftrightarrow y=\frac{1}{\ln x+1} \Leftrightarrow \ln x+1=\frac{1}{y}$ $\quad y\neq 0$

$\ln x=\frac{1}{y}-1 \Leftrightarrow x=e^{\frac{1}{y}-1}$

$h^{-1}(x)=e^{\frac{1}{x}-1}$

$D_{h^{-1}}=\mathbb{R}^*$.

Θεμα 2

• $f(x) = \dfrac{\alpha x+3}{x-1}$ $\quad D_f = \mathbb{R} - \{1\}$

• $f(2) = 5 \Leftrightarrow \dfrac{2\alpha+3}{1} = 5 \Leftrightarrow 2\alpha = 2 \Leftrightarrow \alpha = 1.$

ⓐ νδο f αντιστρεφεται

Εστω $f(x_1) = f(x_2) \Leftrightarrow \dfrac{x_1+3}{x_1-1} = \dfrac{x_2+3}{x_2-1}$

$\Leftrightarrow (x_1+3)(x_2-1) = (x_2+3)(x_1-1)$

$\cancel{x_1 x_2} - x_1 + 3x_2 - \cancel{3} = \cancel{x_1 x_2} - x_2 + 3x_1 - \cancel{3}$

$4x_2 = 4x_1$

$x_1 = x_2$

Αφου f 1-1 αντιστρεφεται.

ⓑ νδο f και f^{-1} ειναι ισες.

$f(x) = y \Leftrightarrow y = \dfrac{x+3}{x-1} \Leftrightarrow y(x-1) = x+3 \Leftrightarrow yx - y = x+3$

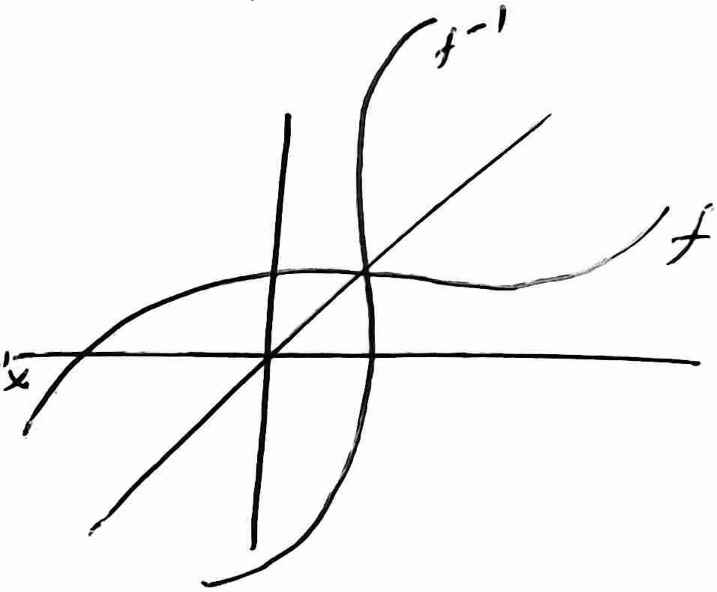
$yx - x = y+3$

$x(y-1) = y+3$ $\qquad$ Ισα ✓.

$x = \dfrac{y+3}{y-1}, \ y \neq 1$

$$\boxed{f^{-1}(x) = \frac{x+3}{x-1} \quad D_{f^{-1}} = \mathbb{R} - \{1\}}$$

(γ). Κοινά σημεία των $C_f, C_{f^{-1}}$ με τον άξονα συμμετρί-
τους.



Οι $C_f, C_{f^{-1}}$ είναι
συμμετρικές ως προς τον $x'x$

Άρα $f(x)=x$, $f^{-1}(x)=x$
Ισοδύναμα.

$$\frac{x+3}{x-1}=x \iff x+3=x^2-x \iff x^2-2x-3=0$$

$x=3$ $\quad x=-1$

$A(3,3)$ $\quad B(-1,-1)$

(δ). Κυρτότητα $f(x)$.

$$f'(x)=\frac{x-1-x-3}{(x-1)^2}=\frac{-4}{(x-1)^2}$$

$$f''(x)=\frac{4\cdot 2(x-1)}{(x-1)^4}=\frac{8}{(x-1)^3}$$

x		1	
f''	$-$		$+$
f	∩		∪

Θέμα 3

- $f(x) = \ln x$, $D_f = (0, +\infty)$
- $g(x) = \sqrt{1-x}$ $D_g = (-\infty, 1]$

ⓐ Βρείτε $h = f - g$ και $\varphi = g \circ f$.

$h(x) = (f-g)(x) = f(x) - g(x) = \ln x - \sqrt{1-x}$

$D_h = D_f \cap D_g = (0, 1]$

$\varphi(x) = (g \circ f)(x) = g(f(x)) = \sqrt{1 - \ln x}$

πρέπει $x \in D_f$ και $f(x) \in D_g$

$x > 0$ και $\ln x \leq 1$ $D_\varphi = (0, e]$

$x \leq e$

ⓑ Νδο φ αντιστρέφεται και να βρω φ^{-1}

- $x_1 < x_2 \Rightarrow 1 - \ln x_1 > 1 - \ln x_2 \Rightarrow \sqrt{1 - \ln x_1} > \sqrt{1 - \ln x_2}$

$f(x_1) > f(x_2)$

άρα $f \downarrow$ άρα αντιστρέφεται

Έστω $y = \sqrt{1 - \ln x}$ $y \geq 0$

$y^2 = 1 - \ln x$

$\ln x = 1 - y^2$

$x = e^{1-y^2}$

$$\boxed{\begin{array}{c} \varphi^{-1}(x) = e^{1-x^2} \\ D_{\varphi^{-1}} = [0, +\infty) \end{array}}$$

(γ) νδο η $\varphi^{-1} \downarrow$.

Εστω $\varphi^{-1}(x_1) < \varphi^{-1}(x_2)$

$\varphi \downarrow$

$\varphi(\varphi^{-1}(x_1)) > \varphi(\varphi^{-1}(x_2))$

$x_1 > x_2$ αρα $\varphi^{-1} \downarrow$

(δ) εξισωση $\ln \varepsilon\varphi x = \sqrt{1-\eta\mu x} - \sqrt{1-\sigma\upsilon\nu x}$ $\left(0, \frac{\pi}{2}\right)$

$\ln \frac{\eta\mu x}{\sigma\upsilon\nu x} = \sqrt{1-\eta\mu x} - \sqrt{1-\sigma\upsilon\nu x}$

$\ln \eta\mu x - \ln \sigma\upsilon\nu x = \sqrt{1-\eta\mu x} - \sqrt{1-\sigma\upsilon\nu x}$

$\ln \eta\mu x - \sqrt{1-\eta\mu x} = \ln \sigma\upsilon\nu x - \sqrt{1-\sigma\upsilon\nu x}$

$h(\eta\mu x) = h(\sigma\upsilon\nu x)$

h ος 1-1

$\eta\mu x = \sigma\upsilon\nu x \Rightarrow \frac{\eta\mu x}{\sigma\upsilon\nu x} = 1 \Leftrightarrow \varepsilon\varphi x = 1$

$x = \frac{\pi}{4}$

Μονοτονια h

$h(x) = \ln x - \sqrt{1-x}$ $D_h = (0, 1]$

$h'(x) = \frac{1}{x} + \frac{1}{2\sqrt{1-x}} > 0$

$h \nearrow$ αρα 1-1.

Θεμα 4

- $f(x) = x^3 + x - 10$

(α) Εφαπτομενη στο $x_0 = 2$

$\varepsilon: y - f(2) = f'(2)(x-2) \Leftrightarrow y - 0 = 13(x-2)$

$\boxed{y = 13x - 26}$

- $f(2) = 2^3 + 2 - 10 = 0$
- $f'(x) = 3x^2 + 1 > 0$ $\quad f \nearrow$
- $f'(2) = 13$

(β) Μονοτονια - κυρτοτητα.

$f'(x) = 3x^2 + 1 > 0 \Leftrightarrow f \nearrow$

$f''(x) = 6x$

$\rightarrow 6x = 0 \Leftrightarrow \boxed{x = 0}$

x		0	
f''	$-$	0	$+$
f	$\frown$		$\smile$

$A(0, -10)$ Σ.Κ.

(γ) Ανισωση $\quad x > \dfrac{10}{x^2+1}$.

$x^3 + x > 10$

$x^3 + x - 10 > 0$

$f(x) > 0$

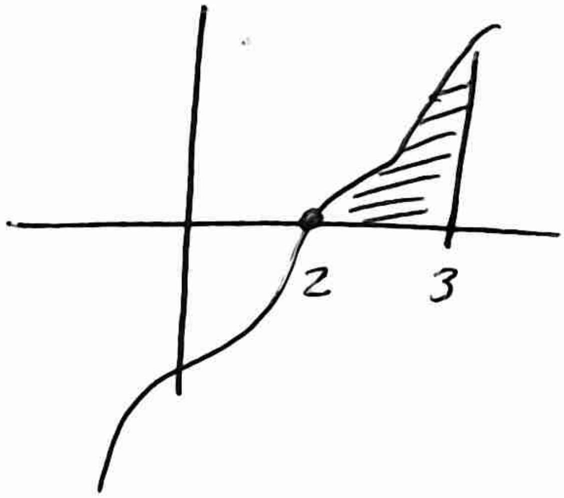
$f(x) > f(2)$

$f \nearrow$

$x > 2$

(5) $E: G_f, x'x, x=3$

$$E = \int_2^3 |f(x)|\, dx = $$



P.T.U $f(x)$

$f(x)=0$

$f(x)=f(2)$

$f: 1-1$

$x=2$

$$= \int_2^3 x^3 + x - 10 \; dx =$$

$$= \int_2^3 x^3\, dx + \int_2^3 x\, dx - \int_2^3 10\, dx$$

$$= \frac{1}{4}\left(x^4\right)_2^3 + \frac{1}{2}\left(x^2\right)_2^3 - 10\left(x\right)_2^3$$

$$= \frac{1}{4}(81-16) + \frac{1}{2}(9-4) - 10$$

$$= \frac{65}{4} + \frac{5}{2} - 10 = \frac{65}{4} + \frac{10}{4} - \frac{40}{4} =$$

$$= \frac{35}{4}.$$

Θέμα 5

• $f(x) = (x-1)e^x \quad D_f = \mathbb{R}$.

(α) Μονοτονία.

$f'(x) = e^x + (x-1)e^x = e^x(1+x-1) = xe^x$

$\longrightarrow f'(x) = 0 \Leftrightarrow xe^x = 0 \Leftrightarrow \boxed{x=0}$.

x		0	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

(β) Ν.δ.ο. $f(x) \geq -1$. $\forall x \in \mathbb{R}$.

$f(x) \geq f(0) \Rightarrow f(x) \geq -1$. $\forall x \in \mathbb{R}$

(γ). $f''(x) = e^x + xe^x = e^x(x+1)$

x		-1	
f''	$-$	0	$+$
f	$\frown$		$\smile$

$A\left(-1, -\frac{2}{e^2}\right)$.

$y - f(-1) = f'(-1)(x+1)$

$\boxed{y + \frac{2}{e} = -\frac{1}{e}\cdot(x+1)}$

(5) E : $(f, x'x, y'y, x=1$

$$E = \int_0^1 |f(x)|\,dx = \int_0^1 \left|\overset{\ominus}{(x-1)e^x}\right| dx = -\int_0^1 (x-1)e^x\,dx$$

$$E = -\int_0^1 (x-1)(e^x)'\,dx = -\left[\left((x-1)e^x\right)_0^1 - \int_0^1 e^x\,dx\right]$$

$$E = -\left[+1 - (e^x)_0^1\right] = -1 + (e-1) = -1+e-1$$

$$\underline{\underline{= e-2}}$$

Θεμα 6

• $f(x) = x^3 - 3x^2 + 1$

(α) Μονοτονια.

$f'(x) = 3x^2 - 6x$

$\longrightarrow 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0 \quad x = 2$

x		0		2	
f'	+	0	−	0	+
f	↗		↘		↗

(β) Ανισωση $f(3+x^2) > f(2+2x^2)$.

• $3+x^2 > 2$
• $2+2x^2 > 2$

$\Big\}$ Στο $(2, +\infty)$ η f ↗

$3+x^2 > 2+2x^2$

$0 > x^2 - 1$

x		−1		1	
x^2-1	+	0	−	0	+

$x \in (-1, 1)$.

(γ). κυρτότητα στο σημείο καμπής.

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

x		1	
f''	$-$	0	$+$
f	$\frown$		$\smile$

$$\varepsilon : y - f(1) = f'(1)(x-1)$$

$$y - (-1) = -3(x-1)$$

$$y + 1 = -3x + 3 \qquad \underline{\underline{y = -3x + 2}}$$

(δ) $$\int_{-1}^{0} f(x+1)\,dx = \int_{0}^{1} f(t)\,dt =$$

$$\boxed{\begin{array}{ll} x+1 = t & \\ 1 \cdot dx = 1 \cdot dt & \\ x = 0 & t = 1 \\ x = -1 & t = 0 \end{array}}$$

$$= \int_0^1 t^3 - 3t^2 + 1\,dt =$$

$$= \frac{1}{4}\left(t^4\right)_0^1 - \left(t^3\right)_0^1 + \left(t\right)_0^1 =$$

$$= \frac{1}{4} - 1 + 1 = \frac{1}{4}.$$

Θεμα 7

• $f(x) = \dfrac{x^2+1}{e^x}$

(α) Μονοτονια $f(x)$

$$f'(x) = \frac{2x e^x - (x^2+1)e^x}{e^{2x}} = \frac{2x - x^2 - 1}{e^x} = \frac{-(x-1)^2}{e^x} \le 0$$

$f \downarrow$

(β) Ανισωση $x^2+1 > e^x$

$$\frac{x^2+1}{e^x} > 1 \iff f(x) > 1 \Rightarrow f(x) > f(0) \underset{f\downarrow}{\Rightarrow} \underline{\underline{x < 0}}$$

(γ) εξισωση $\ln(x^2+1) = x$.

$$x^2+1 = e^x$$

$$\frac{x^2+1}{e^x} = 1 \iff f(x) = f(0) \underset{f \text{ 1-1}}{\iff} \underline{\underline{x = 0}}$$

(8) Ν.δ.ο. f αντιστρέφεται και $D_{f^{-1}}$

Αφού $f \downarrow \Rightarrow f$ 1-1 άρα αντιστρέφεται.

$$D_{f^{-1}} = \Sigma T_f$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+1}{e^x} = \lim_{x\to-\infty} (x^2+1)\frac{1}{e^x} = (+\infty)(+\infty) = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+1}{e^x} = \lim_{x\to+\infty} \frac{2x}{e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = 0.$$

$$\Sigma T_f = (0, +\infty)$$

$$D_{f^{-1}} = (0, +\infty).$$

Θεμα 8

• $f(x) = x^5 + 2x - 3$

ⓐ Νδο f αντιστρεψεται και $D_{f^{-1}}$

$f'(x) = 5x^4 + 2 > 0 \Rightarrow f \nearrow$ αρα αντιστρεψεται

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^5 + 2x - 3 = \lim_{x \to -\infty} x^5 = -\infty$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^5 + 2x - 3 = +\infty$

$\Sigma T_f = \mathbb{R}$ αρα $D_{f^{-1}} = \mathbb{R}$

ⓑ $f(2x-1) + 3 - 2x > x^5$

$f(2x-1) > x^5 + 2x - 3$

$f(2x-1) > f(x)$

$f \nearrow$

$2x - 1 > x$

$x > 1$.

⑧ $f\left(1+f^{-1}(x^2-7x)\right)=-3$

$f\left(1+f^{-1}(x^2-7x)\right)=f(0)$

$f(3)-)$

$1+f^{-1}(x^2-7x)=0$

$f^{-1}(x^2-7x)=-1$

$f\left(f^{-1}(x^2-7x)\right)=f(-1)$

$x^2-7x=-6$

$x^2-7x+6=0$

$x=1$ $\quad$ $x=6$.

⑧ $\int_1^e \frac{f(\ln x)}{x}\,dx = \int_0^1 f(t)\,dt =$

$\ln x = t$

$\frac{1}{x}dx = dt$

$=\int_0^1 t^5+2t-3 \quad dt$

$=\frac{1}{6}(t^6)_0^1+(t^2)_0^1-3(t)_0^1$

$=\frac{1}{6}+1-3 \quad =\frac{1}{6}-2$

Θεμα 9

- $f(x) = \ln(x-1)$ $\qquad D_f = (1, +\infty)$
- $g(x) = \frac{1}{x}$ $\qquad D_g = \mathbb{R}^*$

(α) $h = f \circ g$.

$h(x) = (f \circ g)(x) = f(g(x)) = \ln\left(\frac{1}{x} - 1\right) = \ln\left(\frac{1-x}{x}\right)$

$x \in D_g$ και $g(x) \in D_f$

$x \neq 0$ $\qquad \frac{1}{x} > 1 \Leftrightarrow \frac{1-x}{x} > 0$

x		0		1	
$1-x$	+		+	0	−
x	−	0	+		+
$\frac{1-x}{x}$	−		+		−

$x \in (0,1)$.

$D_h = (0,1)$

(β) Νδο h 1-1

$h(x_1) = h(x_2) \Leftrightarrow \ln\left(\frac{1-x_1}{x_1}\right) = \ln\left(\frac{1-x_2}{x_2}\right)$

$\frac{1-x_1}{x_1} = \frac{1-x_2}{x_2}$ $\qquad \Leftrightarrow (1-x_1)x_2 = (1-x_2)x_1$

$x_2 - x_1x_2 = x_1 - x_1x_2 \Leftrightarrow x_1 = x_2$.

(γ) · Αφού h 1-1 αντιστρέψιμη

$h(x)=y \Leftrightarrow y=\ln\left(\frac{1-x}{x}\right) \Leftrightarrow e^y=\frac{1-x}{x}$

$\Leftrightarrow e^y x=1-x \Leftrightarrow x+e^y x=1 \Leftrightarrow x(1+e^y)=1$

$x=\frac{1}{1+e^y}$ πρέπει $1+e^y\neq 0$ που ισχύει.

$h^{-1}(x)=\frac{1}{1+e^x}$

$D_{h^{-1}}=\mathbb{R}$.

(δ) · $\int_0^1 e^x h^{-1}(x)\,dx=\int_0^1 \frac{e^x}{1+e^x}\,dx$

$=\left[\ln(1+e^x)\right]_0^1=\ln(e+1)-\ln 2=$

$=\ln\left(\frac{e+1}{2}\right).$

Θεμα 10

• $f(x) = x\ln x - x + 3$ $\qquad D_f = (0,+\infty)$

(α) $f'(x) = \ln x + 1 - 1 = \ln x$

x		1	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

$f'(x) = \ln x.$

$\longrightarrow \ln x = 0 \iff x = 1$

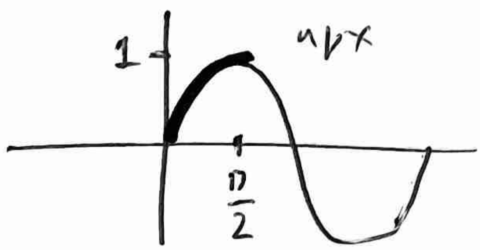
$f(x) \geq f(1)$

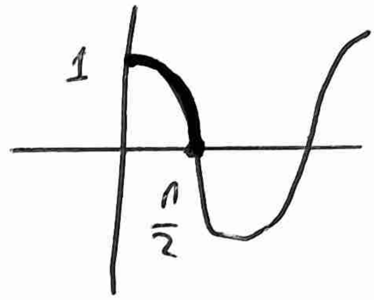
$f(x) \geq 2.$

(β) $\longrightarrow$

(γ) $f(\eta\mu x) = f(\sigma\upsilon\nu x)$. στο $\left(0, \frac{\pi}{2}\right)$.

Οταν $x \in \left(0, \frac{\pi}{2}\right)$ το $\eta\mu x \in (0,1)$ και $\sigma\upsilon\nu x \in (0,1)$.





Στο $(0,1)$ η $f \downarrow$ αρα 1−1

$\eta\mu x = \sigma\upsilon\nu x$

$\frac{\eta\mu x}{\sigma\upsilon\nu x} = 1$

$\varepsilon\varphi x = 1 \qquad x = \frac{\pi}{4}.$

(δ) $f(x) = x\ln x - x + 3$, $x > 0$

$f'(x) = \ln x$

$f''(x) = \frac{1}{x} > 0$ αρα f κυρτη.

Οταν η $f(x)$ ειναι κυρτη, η $f(x)$ ειναι πανω απο οποιαδηποτε εφαπτομενη στο τυχαιο $x_0 \in D_f$. εκτος του σημειου επαφης.

ΔΝδ $f(x) \geq \alpha x + \beta$

με το " = " για $x = x_0$.