Άσκηση 1

Να υπολογιστούν τα όρια.

(α) $\lim\limits_{x\to 3} \dfrac{x^3-5x^2+4x-6}{x^2-4x+5} = \dfrac{3^3-5\cdot 3^2+4\cdot 3-6}{3^2-4\cdot 3+5} = -6$

(β) $\lim\limits_{x\to -2} \dfrac{x^3+4x^2+x-6}{x^2-2x-8} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to -2} \dfrac{(x+2)(x^2+2x-3)}{(x+2)(x-4)}$

$= \lim\limits_{x\to -2} \dfrac{x^2+2x-3}{x-4} = \dfrac{1}{2}$

(γ) $\lim\limits_{x\to 2} \dfrac{\sqrt{x^2+5}-3}{x^2-2x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 2} \dfrac{(\sqrt{x^2+5}-3)(\sqrt{x^2+5}+3)}{(x^2-2x)(\sqrt{x^2+5}+3)}$

$= \lim\limits_{x\to 2} \dfrac{\sqrt{x^2+5}^{\,2}-3^2}{x(x-2)(\sqrt{x^2+5}+3)} = \lim\limits_{x\to 2} \dfrac{x^2-4}{x(x-2)(\sqrt{x^2+5}+3)}$

$= \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{x\cancel{(x-2)}(\sqrt{x^2+5}+3)} = \dfrac{4}{2\cdot 6} = \dfrac{2}{6} = \dfrac{1}{3}$

(δ) $$\lim_{x\to 1}\frac{\sqrt{x+3}-\sqrt{5-x}}{\sqrt{2x-1}-1}=$$

$$=\lim_{x\to 1}\frac{(\sqrt{x+3}-\sqrt{5-x})(\sqrt{x+3}+\sqrt{5-x})(\sqrt{2x-1}+1)}{(\sqrt{2x-1}-1)(\sqrt{2x-1}+1)(\sqrt{x+3}+\sqrt{5-x})}$$

$$=\lim_{x\to 1}\frac{(\sqrt{x+3}^2-\sqrt{5-x}^2)(\sqrt{2x-1}+1)}{(\sqrt{2x-1}^2-1^2)(\sqrt{x+3}+\sqrt{5-x})}$$

$$=\lim_{x\to 1}\frac{(x+3-5+x)(\sqrt{2x-1}+1)}{(2x-2)(\sqrt{x+3}+\sqrt{5-x})}=\lim_{x\to 1}\frac{(2x-2)(\sqrt{2x-1}+1)}{(2x-2)(\sqrt{x+3}+\sqrt{5-x})}$$

$$=\lim_{x\to 1}\frac{\sqrt{2x-1}+1}{\sqrt{x+3}+\sqrt{5-x}}=\frac{2}{2+2}=\frac{2}{4}=\frac{1}{2}.$$

(ε) $$\lim_{x\to 1}\frac{|\overset{\ominus}{x-3}|-|\overset{\oplus}{x^2-7x+9}|+4-3x}{x^2-x}=$$

$$=\lim_{x\to 1}\frac{-x+3-(x^2-7x+9)+4-3x}{x^2-x}=$$

$$=\lim_{x\to 1}\frac{-x^2+3x-2}{x^2-x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 1}\frac{-\cancel{(x-1)}(x-2)}{x\cancel{(x-1)}}=1.$$

(3) $\lim\limits_{x\to 1} \dfrac{|x^2-x| + |-x^2-2x+3| + x^3-1}{x^2-1-|1-x|}$

Το οριο δεν υπαρχει!

x		-3		0		1	
x^2-x	$+$		$+$	0	$-$	0	$+$
$-x^2-2x+3$	$-$	0	$+$		$+$	0	$-$
$1-x$	$-$		$+$		$-$	0	$-$

Με ενδιαφερει η περιοχη κοντα στο 1.

$\rightarrow \lim\limits_{x\to 1^-} \dfrac{|x^2-x| + |-x^2-2x+3| + x^3-1}{x^2-1-|1-x|} =$

$= \lim\limits_{x\to 1^-} \dfrac{-x^2+x-x^2-2x+3+x^3-1}{x^2-1+1-x} = \lim\limits_{x\to 1^-} \dfrac{x^3-2x^2-x+2}{x^2-x}$

$= \lim\limits_{x\to 1^-} \dfrac{x^2(x-2)-(x-2)}{x(x-1)} = \lim\limits_{x\to 1^-} \dfrac{(x-2)\cancel{(x-1)}(x+1)}{x\cancel{(x-1)}} = -2$

$\rightarrow \lim\limits_{x\to 1^+} \dfrac{|x^2-x| + |-x^2-2x+3| + x^3-1}{x^2-1-|1-x|} =$

$= \lim\limits_{x\to 1^+} \dfrac{x^2-x+x^2+2x-3+x^3-1}{x^2-1+1-x} =$

$= \lim\limits_{x\to 1^+} \dfrac{x^3+2x^2+x-4}{x^2-x} = \lim\limits_{x\to 1^+} \dfrac{\cancel{(x-1)}(x^2+3x+4)}{x\cancel{(x-1)}}$

$= 8.$

(η) $\lim_{x\to 0} \dfrac{\eta\mu^2 x - 2\eta\mu x}{x^2+x} = \lim_{x\to 0} \dfrac{\frac{\eta\mu x}{x}\,\eta\mu x - 2\,\frac{\eta\mu x}{x}}{x+1} =$

$= \dfrac{1\cdot 0 - 2\cdot 1}{0+1} = \dfrac{-2}{1} = -2$

(θ) $\lim_{x\to 0} \dfrac{\sigma\upsilon\nu x - 1}{\sqrt{x^2+x+4} - 2} = \lim_{x\to 0} \dfrac{(\sigma\upsilon\nu x - 1)(\sqrt{x^2+x+4}+2)}{(\sqrt{x^2+x+4}-2)(\sqrt{x^2+x+4}+2)}$

$= \lim_{x\to 0} \dfrac{(\sigma\upsilon\nu x - 1)(\sqrt{x^2+x+4}+2)}{\sqrt{x^2+x+4}^{\,2} - 2^2} = \lim_{x\to 0} \dfrac{(\sigma\upsilon\nu x - 1)(\sqrt{x^2+x+4}+2)}{x^2+x}$

$= \lim_{x\to 0} \dfrac{(\sigma\upsilon\nu x - 1)(\sqrt{x^2+x+4}+2)}{x(x+1)} = \lim_{x\to 0} \dfrac{\sigma\upsilon\nu x - 1}{x} \cdot \dfrac{\sqrt{x^2+x+4}+2}{x+1}$

$= 0 \cdot \dfrac{4}{1} = 4.$

(ι) $\lim_{x\to 0} \dfrac{1-\sigma\upsilon\nu x}{x^2} = \lim_{x\to 0} \dfrac{(1-\sigma\upsilon\nu x)(1+\sigma\upsilon\nu x)}{x^2(1+\sigma\upsilon\nu x)} =$

$= \lim_{x\to 0} \dfrac{1-\sigma\upsilon\nu^2 x}{x^2(1+\sigma\upsilon\nu x)} = \lim_{x\to 0} \dfrac{\eta\mu^2 x}{x^2(1+\sigma\upsilon\nu x)} =$

$= \lim_{x\to 0} \dfrac{\eta\mu^2 x}{x^2} \cdot \dfrac{1}{1+\sigma\upsilon\nu x} = 1 \cdot \dfrac{1}{2} = \dfrac{1}{2}$

(κ) $\lim_{x\to 0} (x^2+x)\,\sigma\upsilon\nu\frac{1}{x}$

x	-1	0
x^2+x	+ 0 −	0 +

Ισχυει οτι $-1 \le \sigma\upsilon\nu\frac{1}{x} \le 1$

1. Αν $x<0$ τοτε $-x^2-x \geq (x^2+x)\,\sigma\upsilon\nu\frac{1}{x} \geq x^2+x$

$\left.\begin{array}{l} \cdot \lim_{x\to 0^-}(-x^2-x)=0 \\ \cdot \lim_{x\to 0^-}(x^2+x)=0 \end{array}\right\}$ Απο Κ.Π $\lim_{x\to 0^-}(x^2+x)\,\sigma\upsilon\nu\frac{1}{x}=0$

2. Αν $x>0$ τοτε $-x^2-x \le (x^2+x)\,\sigma\upsilon\nu\frac{1}{x} \le x^2+x$

$\left.\begin{array}{l} \cdot \lim_{x\to 0^+}(-x^2-x)=0 \\ \cdot \lim_{x\to 0^+}(x^2+x)=0 \end{array}\right\}$ Απο Κ.Π $\lim_{x\to 0^+}(x^2+x)\,\sigma\upsilon\nu\frac{1}{x}=0.$

Αρα $\lim_{x\to 0}(x^2+x)\,\sigma\upsilon\nu\frac{1}{x}=0.$

(λ) $\lim_{x\to 0}\frac{\eta\mu 5x}{x} = \lim_{x\to 0} 5\,\frac{\eta\mu 5x}{5x} = 5\cdot 1 = 5.$

(μ) $\lim_{x\to 3}\frac{\sqrt[3]{x-2}+3\sqrt[4]{x-2}-4}{\sqrt[3]{x-2}-\sqrt[6]{x-2}} = \lim_{\kappa\to 1}\frac{\kappa^4+3\kappa^3-4}{\kappa^4-\kappa^2}$

Θετω $\sqrt[12]{x-2}=\kappa$
αρα $\sqrt[3]{x-2}=\kappa^4$ $\quad \sqrt[6]{x-2}=\kappa^2$
$\sqrt[4]{x-2}=\kappa^3$ $\quad x\to 3$, $\kappa\to 1$

$= \lim_{\kappa\to 1}\frac{(\kappa-1)(\kappa^3+4\kappa^2+4\kappa+4)}{\kappa^2(\kappa-1)(\kappa+1)}$

$= \frac{13}{2}$

(v) $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{\ln x^x} = \lim_{x\to 0} e^{x\ln x} = 1.$

$\longrightarrow \lim_{x\to 0}\left[x\ln x\right] = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -\frac{x^2}{x} = 0$

Ασκηση 2

Να υπολογιστουν τα παρακατω όρια

(α) $\lim\limits_{x \to 0} \dfrac{1}{x^2} = +\infty$

(β) $\lim\limits_{x \to 0} \dfrac{1}{x^3}$ Το οριο δεν υπαρχει

x	0
x^3	$-$ 0 $+$

$\to \lim\limits_{x \to 0^-} \dfrac{1}{x^3} = -\infty$

$\to \lim\limits_{x \to 0^+} \dfrac{1}{x^3} = +\infty$

(γ) $\lim\limits_{x \to 2} \dfrac{1}{x^2-4x+4} = \lim\limits_{x \to 2} \dfrac{1}{(x-2)^2} = +\infty.$

(δ) $\lim\limits_{x \to -3} \dfrac{1}{|x+3|} = +\infty$

(ε) $\lim\limits_{x \to 0} \dfrac{1}{\sigma\upsilon\nu x - 1} = -\infty$

Ισχυει οτι $\sigma\upsilon\nu x \leq 1 \Leftrightarrow \sigma\upsilon\nu x - 1 \leq 0$

(ζ) $\lim\limits_{x \to 4} \dfrac{1}{4-x}$ Το οριο δεν υπαρχει

→ $\lim\limits_{x \to 4^-} \dfrac{1}{4-x} = +\infty$

x	4
$4-x$	+ 0 −

→ $\lim\limits_{x \to 4^+} \dfrac{1}{4-x} = -\infty$

(η) $\lim\limits_{x \to 3} \dfrac{x-5}{x^3-6x^2+9x} = \lim\limits_{x \to 3} \dfrac{x-5}{x(x-3)^2} = \lim\limits_{x \to 3} \dfrac{x-5}{x} \cdot \dfrac{1}{(x-3)^2}$

$= \dfrac{-2}{3} \cdot +\infty = -\infty$

(θ) $\lim\limits_{x \to 2} \dfrac{x-6}{x^2-x-2} = \lim\limits_{x \to 2} \dfrac{x-6}{x+1} \cdot \dfrac{1}{x-2}$

→ $\lim\limits_{x \to 2^-} \dfrac{x-6}{x+1} \cdot \dfrac{1}{x-2} = \dfrac{-4}{3} \cdot (-\infty) = +\infty$

x	2
$x-2$	− 0 +

→ $\lim\limits_{x \to 2^+} \dfrac{x-6}{x+1} \cdot \dfrac{1}{x-2} = -\dfrac{4}{3} (+\infty) = -\infty$

Το οριο δεν υπαρχει!.

x	0
x	− 0 +
ημx	− 0 +

(ι) $\lim\limits_{x \to 0} \dfrac{1}{x\,ημx} = +\infty$

· $\lim\limits_{x \to 0^-} \dfrac{1}{x} \cdot \dfrac{1}{ημx} = (-\infty)(-\infty) = +\infty$

· $\lim\limits_{x \to 0^+} \dfrac{1}{x} \cdot \dfrac{1}{ημx} = (+\infty)(+\infty) = +\infty$

Ασκηση 3

Να υπολογιστουν τα ορια

(α) $\lim\limits_{x \to +\infty} (x^4) = +\infty$

(β) $\lim\limits_{x \to -\infty} (x^3) = -\infty$

(γ) $\lim\limits_{x \to -\infty} (x^6) = +\infty$

(δ) $\lim\limits_{x \to +\infty} \left(\frac{1}{x^5}\right) = 0$

(ε) $\lim\limits_{x \to +\infty} (2x^3 - 5x^2 + 7x - 6) = \lim\limits_{x \to +\infty} (2x^3) = +\infty$

(ζ) $\lim\limits_{x \to +\infty} \frac{6x^3 - 3x^2 + 5}{2x^3 + 7x - 6} = \lim\limits_{x \to +\infty} \frac{6x^3}{2x^3} = \lim\limits_{x \to +\infty} \frac{6}{2} = 3$

(η) $\lim\limits_{x \to +\infty} \frac{\overset{+}{|x^4 + 6x^3 - 5x + 6|} - x^4}{\underset{+}{|4x^3 + 2x^2 - 3x|} + x^3} = \lim\limits_{x \to +\infty} \frac{x^4 + 6x^3 - 5x + 6 - x^4}{4x^3 + 2x^2 - 3x + x^3}$

$= \lim\limits_{x \to +\infty} \frac{6x^3 - 5x + 6}{5x^3 + 2x^2 - 3x} = \lim\limits_{x \to +\infty} \frac{6x^3}{5x^3} = \frac{6}{5}.$

θ) $\lim_{x \to +\infty} \left[\sqrt{x^2-3x+5} + x\right] = +\infty$

i) $\lim_{x \to +\infty} \left[\sqrt{x^2-x+2} - 3x\right] = \lim_{x \to +\infty} \left[\sqrt{x^2\left(1-\frac{1}{x}+\frac{2}{x^2}\right)} - 3x\right]$

$= \lim_{x \to +\infty} \left[\overset{+}{|x|}\sqrt{1-\frac{1}{x}+\frac{2}{x^2}} - 3x\right] = \lim_{x \to +\infty} \left[x\sqrt{1-\frac{1}{x}+\frac{2}{x^2}} - 3x\right]$

$= \lim_{x \to +\infty} \left[x\left(\sqrt{1-\frac{1}{x}+\frac{2}{x^2}} - 3\right)\right] = +\infty(1-3) = -\infty.$

k) $\lim_{x \to -\infty} \left[\sqrt{4x^2-5x+2} + 2x\right] = \lim_{x \to -\infty} \frac{\left(\sqrt{4x^2-5x+2}+2x\right)\left(\sqrt{4x^2-5x+2}-2x\right)}{\sqrt{4x^2-5x+2} - 2x}$

$= \lim_{x \to -\infty} \frac{\sqrt{4x^2-5x+2}^{\,2} - (2x)^2}{\sqrt{4x^2-5x+2} - 2x} = \lim_{x \to -\infty} \frac{4x^2-5x+2-4x^2}{\sqrt{4x^2-5x+2} - 2x}$

$= \lim_{x \to -\infty} \frac{-5x+2}{-x\sqrt{4-\frac{5}{x}+\frac{2}{x^2}} - 2x} = \lim_{x \to -\infty} \frac{x\left(-5+\frac{2}{x}\right)}{-x\left(\sqrt{4-\frac{5}{x}+\frac{2}{x^2}} + 2\right)}$

$= \frac{-5}{-4} = \frac{5}{4}$

(λ) $\lim_{x\to+\infty} \frac{5^x}{3^x} = \lim_{x\to+\infty} \left(\frac{5}{3}\right)^x = +\infty$

(μ) $\lim_{x\to+\infty} \frac{2^x}{7^x} = \lim_{x\to+\infty} \left(\frac{2}{7}\right)^x = 0$

(ν) $\lim_{x\to+\infty} \frac{3^x + 2^{x+1}}{3^{x+2}+2^x} = \lim_{x\to+\infty} \frac{3^x + 2^x \cdot 2}{3^x \cdot 9 + 2^x} =$

$= \lim_{x\to+\infty} \frac{3^x\left(1 + 2\cdot\left(\frac{2}{3}\right)^x\right)}{3^x\left(9+\left(\frac{2}{3}\right)^x\right)} = \frac{1}{9}$

(ξ) $\lim_{x\to-\infty} \left(\frac{5^{x+1}+2^{x+3}}{5^x - 2^{x+1}}\right) = \lim_{x\to-\infty}\left(\frac{5^x\cdot 5 + 2^x \cdot 8}{5^x - 2^x \cdot 2}\right)$

$= \lim_{x\to-\infty} \frac{2^x\left(\left(\frac{5}{2}\right)^x\cdot 5 + 8\right)}{2^x\left(\left(\frac{5}{2}\right)^x - 2\right)} = \frac{8}{-2} = -4$

(ο) $\lim_{x\to 0^+} \frac{6\ln^2 x - 5\ln x + 4}{3\ln^2 x + 2\ln x - 1} = \lim_{k\to-\infty} \frac{6k^2 - 5k + 4}{3k^2+2k-1} =$

Θέτω $\ln x = k$ Όταν $x \to 0^+$ τότε $k \to -\infty$

$= \lim_{k\to-\infty} \frac{6k^2}{3k^2} = 2.$

(π) $\lim_{x\to+\infty} \left[\ln(x^3-5x) - 2\ln(x+2) \right] =$

$= \lim_{x\to+\infty} \left[\ln(x^3-5x) - \ln(x+2)^2 \right] =$

$= \lim_{x\to+\infty} \ln\left(\frac{x^3-5x}{(x+2)^2} \right) = \lim_{x\to+\infty} \ln\left(\frac{x^3-5x}{x^2+4x+4} \right)$

$= \lim_{k\to+\infty} \ln k = +\infty.$

$$\boxed{\begin{array}{l} \text{Θέτω } \frac{x^3-5x}{x^2+4x+4} = k \\ \text{όταν } x\to+\infty \text{ το } k\to+\infty \\ \text{γιατί } \lim_{x\to+\infty}\left(\frac{x^3-5x}{x^2+4x+4}\right) = +\infty \end{array}}$$

(ρ) $\lim_{x\to+\infty} \left[\ln(e^x+1) - x \right] = \lim_{x\to+\infty} \ln(e^x+1) - \ln e^x$

$= \lim_{x\to+\infty} \ln\left(\frac{e^x+1}{e^x} \right) = 0.$

$\to \lim_{x\to+\infty} \frac{e^x+1}{e^x} = \lim_{x\to+\infty} \frac{e^x\left(1+\frac{1}{e^x}\right)}{e^x} = 1.$

σ) $\lim_{x\to+\infty}\left(x\,\eta\mu\frac{1}{x}\right)=\lim_{x\to+\infty}\frac{\eta\mu\frac{1}{x}}{\frac{1}{x}}=1.$

επειδή $\lim_{x\to+\infty}\frac{1}{x}=0$

τ) $\lim_{x\to+\infty}\frac{x\,\eta\mu x}{x^2+1}.$

Ισχύει ότι $-1\le\eta\mu x\le1 \iff -\frac{1}{x^2+1}\le\frac{\eta\mu x}{x^2+1}\le\frac{1}{x^2+1}$

$\iff -\frac{x}{x^2+1}\le\frac{x\,\eta\mu x}{x^2+1}\le\frac{x}{x^2+1}$

$\left.\begin{array}{l}\cdot\ \lim_{x\to+\infty}-\frac{x}{x^2+1}=0\\ \cdot\ \lim_{x\to+\infty}\frac{x}{x^2+1}=0\end{array}\right\}$ Από κριτήριο Παρεμβολής $\lim_{x\to+\infty}\frac{x\,\eta\mu x}{x^2+1}=0.$

υ) $\lim_{x\to+\infty}\ln\left(\frac{2-\eta\mu x}{x}\right)=-\infty.$

Ισχύει ότι $-1\le\eta\mu x\le1 \iff 1\ge-\eta\mu x\ge-1 \iff$

$3\ge2-\eta\mu x\ge1 \iff \frac{3}{x}\ge\frac{2-\eta\mu x}{x}\ge\frac{1}{x}$

$\left.\begin{array}{l}\lim_{x\to+\infty}\frac{3}{x}=0\\ \lim_{x\to+\infty}\frac{1}{x}=0\end{array}\right\}$ Από Κ.Π $\lim_{x\to+\infty}\frac{2-\eta\mu x}{x}=0$

(φ) $\lim_{x\to+\infty} x^x = \lim_{x\to+\infty} e^{\ln x^x} = \lim_{x\to+\infty} e^{x\ln x} = +\infty$

$\longrightarrow \lim_{x\to+\infty} x\ln x = +\infty$

(χ) $\lim_{x\to-\infty} x\left(\sigma\upsilon\nu\frac{1}{x}-1\right) = \lim_{x\to-\infty} \dfrac{\sigma\upsilon\nu\frac{1}{x}-1}{\frac{1}{x}} = 0$

αφου $\lim_{x\to-\infty} \frac{1}{x} = 0$

(ψ) $\lim_{x\to+\infty} (x+\sigma\upsilon\nu x)$

Ισχυει οτι $-1 \le \sigma\upsilon\nu x \le 1 \Leftrightarrow x-1 \le \sigma\upsilon\nu x + x \le x+1$.

- $\lim_{x\to+\infty} (x-1) = +\infty$
- $\lim_{x\to+\infty} (x+1) = +\infty$

Απο κριτηριο παρεμβολης $\lim_{x\to+\infty} (\sigma\upsilon\nu x + x) = +\infty$.

(ω) $\lim_{x\to+\infty} \dfrac{e^x}{e^x+\eta\mu x} = 1$.

$-1 \le \eta\mu x \le 1 \Leftrightarrow e^x-1 \le e^x+\eta\mu x \le e^x+1$

$\Leftrightarrow \dfrac{1}{e^x-1} \ge \dfrac{1}{e^x+\eta\mu x} \ge \dfrac{1}{e^x+1} \Leftrightarrow \dfrac{e^x}{e^x-1} \ge \dfrac{e^x}{e^x+\eta\mu x} \ge \dfrac{e^x}{e^x+1}$

Μηδενικη × φραγμενη

Μια ειδικη περιπτωση οριου ειναι η παρακατω

- $\lim\limits_{x\to x_0} f(x)=0$ και $\lim\limits_{x\to x_0} g(x)=\infty$

Εστω οτι θελω να υπολογισω το $\lim\limits_{x\to x_0} f(x)\eta\mu g(x)$

Τοτε $-1 \leq \eta\mu g(x) \leq 1$

$$|\eta\mu g(x)| \leq 1$$

$$|f(x)|\,|\eta\mu g(x)| \leq 1\cdot |f(x)|$$

$$|f(x)\eta\mu g(x)| \leq |f(x)|$$

$$-|f(x)| \leq f(x)\eta\mu g(x) \leq |f(x)|$$

$$\left.\begin{array}{l} \to \lim\limits_{x\to x_0} -|f(x)| = 0 \\ \\ \to \lim\limits_{x\to x_0} |f(x)| = 0 \end{array}\right\} \lim\limits_{x\to x_0} f(x)\eta\mu g(x) = 0$$