4. Αν εχω ολοκληρωματα της μορφης.

$\int xe^x\,dx$, $\int x\,\eta\mu x\,dx$, $\int x\,\sigma\upsilon\nu x\,dx$, $\int x\ln x\,dx$

και παραπλησια δουλευω με την παραγοντικη ολοκληρωση $\boxed{\int_\alpha^\beta [f(x)]' g(x)\,dx = \Big[f(x)g(x)\Big]_\alpha^\beta - \int_\alpha^\beta f(x)g'(x)\,dx}$

5. Αν εχω ολοκληρωματα της μορφης.

$\int e^x\eta\mu x\,dx$, $\int e^x\sigma\upsilon\nu x\,dx$ ή παραπλησια

εκτελω το "κυκλικο" ολοκληρωμα.

6. Αν δεν δουλευει τιποτα απο τα παραπανω λογικα πρεπει να θεσω κατι μεσα στο ολοκληρωμα και να δουλεψω οπως εχουμε πει.

Στρατηγικη επιλυσης ολοκληρωματων

Οταν εχουμε μπροστα μας ενα ολοκληρωμα προσπαθουμε να το επιλυσουμε τηρωντας τα επομενα βηματα.

1. Αρχικα προσπαθω να βρω παραγουσα με το ματι για να τελειωνω.

2. Αν ειναι κλασμα κοιταω μηπως ο αριθμητης ειναι η παραγωγος του παρονομαστη γιατι τοτε η παραγουσα που ψαχνω ειναι "ln|παρονομαστη|"

3. Αν ειναι κλασμα με πολυωνυμα πανω και κατω τοτε:

   (i) Αν ο αριθμητης ειναι μικροτερου βαθμου του παρονομαστη και ο παρονομαστης παραγοντοποιειται σε πρωτοβαθμιους παραγοντες δουλευω με τη μεθοδο των A και B.

   (ii) Αν ο αριθμητης εχει βαθμο μεγαλυτερο ή ισο του παρονομαστη τοτε εκτελω διαιρεση πολυωνυμων και μετα οτι προκυψει.

Ασκηση 1

Να υπολογιστουν τα ολοκληρωματα

(α) $\int_0^{\frac{\pi}{2}} \eta\mu x \, dx = (-\sigma\upsilon\nu x)_0^{\pi/2} = -(\sigma\upsilon\nu x)_0^{\pi/2} = -(\sigma\upsilon\nu \frac{\pi}{2} - \sigma\upsilon\nu 0) =$
$= 1$

(β) $\int_1^{e^2} \frac{1}{x} \, dx = \left[\ln|x|\right]_1^{e^2} = \ln e^2 - \ln 1 = 2 - 0 = 2.$

(γ) $\int_0^1 x^2 \, dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3} (x^3)_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$

(δ) $\int_2^1 \frac{1}{x^5} \, dx = \int_2^1 x^{-5} \, dx = \left(\frac{x^{-4}}{-4}\right)_2^1 =$

$= -\frac{1}{4}\left(\frac{1}{x^4}\right)_2^1 = -\frac{1}{4}\left(1 - \frac{1}{16}\right) = -\frac{1}{4} \frac{15}{4} = -\frac{15}{16}$

(ε) $\int_0^4 x\sqrt{x} \, dx = \int_0^4 x \cdot x^{\frac{1}{2}} \, dx = \int_0^4 x^{3/2} \, dx$

$= \left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^4 = \frac{2}{5}\left(\sqrt{x^5}\right)_0^4 = \frac{2}{5} \left(x^2\sqrt{x}\right)_0^4$

$= \frac{2}{5} 32 = \frac{64}{5}.$

Ασκηση 2

Να υπολογιστουν τα ολοκληρωματα

ⓐ $\int_{\frac{\pi}{2}}^{\pi} 2\sigma\upsilon\nu x + \frac{3}{x} \, dx = \int_{\frac{\pi}{2}}^{\pi} 2\sigma\upsilon\nu x \, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{3}{x} \, dx$

$= 2\int_{\frac{\pi}{2}}^{\pi} \sigma\upsilon\nu x \, dx + 3\int_{\frac{\pi}{2}}^{\pi} \frac{1}{x} \, dx = 2(\eta\mu x)_{\frac{\pi}{2}}^{\pi} + 3\,(\ln x)_{\frac{\pi}{2}}^{\pi}$

$= 2(-1) + 3\left[\ln\pi - \ln\frac{\pi}{2}\right] = -2 + 3\ln\pi - (\ln\pi - \ln 2)$

$= -2 + 3\ln\pi - \ln\pi + \ln 2 = -2 + 2\ln\pi + \ln 2.$

ⓑ $\int_0^1 x(3x-1)\,dx = \int_0^1 3x^2 - x \, dx =$

$= 3\int_0^1 x^2 \, dx - \int_0^1 x \, dx = 3 \cdot \frac{1}{3}(x^3)_0^1 - \frac{1}{2}(x^2)_0^1$

$= 1 - \frac{1}{2} = \frac{1}{2}.$

ⓒ $\int_1^2 \frac{x^3+x^2}{x^2}\,dx = \int_1^2 x+1 \, dx = \int_1^2 x\,dx + \int_1^2 1\,dx$

$= \frac{1}{2}(x^2)_1^2 + (x)_1^2 = \frac{3}{2} + 1 = \frac{5}{2}$

Ασκηση 4

Να υπολογιστουν τα ολοκληρωματα

ⓐ $\int_0^{\pi} \sigma\upsilon\nu 2x \, dx = \left(\frac{\eta\mu 2x}{2}\right)_0^{\pi} = \frac{1}{2}\left(\eta\mu 2x\right)_0^{\pi} =$

$= \frac{1}{2} \cdot 0 = 0$

ⓑ $\int_0^1 e^{2x} \, dx = \left(\frac{e^{2x}}{2}\right)_0^1 = \frac{e^2}{2} - \frac{1}{2} = \frac{e^2-1}{2}$

ⓒ $\int_0^1 \frac{1}{e^{3t}} \, dt = \int_0^1 e^{-3t} dt = \left(\frac{e^{-3t}}{-3}\right)_0^1$

$= -\frac{1}{3}\left(e^{-3} - 1\right)$

ⓓ $\int_0^1 e^{2x-1} \, dx = \left(\frac{e^{2x-1}}{2}\right)_0^1 = \frac{e}{2} - \frac{e^{-1}}{2}$

ⓔ $\int_0^1 \frac{1}{3x+2} \, dx = \frac{1}{3}\left(\ln 3x+2\right)_0^1 = \frac{1}{3}\left[\ln 5 - \ln 2\right]$

(ζ) $\displaystyle\int_0^1 (x-2)^4\,dx = \left(\frac{(x-2)^5}{5}\right)_0^1 = \frac{1}{5}\left((x-2)^5\right)_0^1$

$= \frac{1}{5}(-1+32) = \frac{31}{5}.$

(η) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \left(2\sqrt{x-1}\right)_2^5 = 4-2=2$

(θ) $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx = \frac{1}{2}\left[\ln(x^2+1)\right]_0^1$

$= \frac{1}{2}(\ln 2 - 0) = \frac{\ln 2}{2}$

(ι) $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx = \left(\sqrt{x^2+1}\right)_0^1 = \sqrt{2}-1$

(κ) $\displaystyle\int_0^1 x e^{x^2}\,dx = \left(\frac{1}{2}e^{x^2}\right)_0^1 = \frac{1}{2}\left(e^{x^2}\right)_0^1 = \frac{1}{2}(e-1).$

(λ) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\text{συν}x}{\text{ημ}^2x}\,dx = \left(\frac{-1}{\text{ημ}x}\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}} = -\left(1-\frac{1}{\frac{1}{2}}\right) = 1.$

(ιγ) $\int_0^{\pi/3} \varepsilon\varphi x \, dx = \int_0^{\pi/3} \frac{\eta\mu x}{\sigma\upsilon\nu x} \, dx = \left(-\ln \sigma\upsilon\nu x\right)_0^{\pi/3}$

$= -\left(\ln \frac{1}{2} - 0\right) = -\ln \frac{1}{2} = \ln 2.$

(ιδ) $\int_1^2 \frac{\ln x}{x} \, dx = \left(\frac{1}{2} \ln^2 x\right)_1^2 = \frac{1}{2}\left(\ln^2 x\right)_1^2 = \frac{1}{2} \ln^2 2$

(ιε) $\int_a^{a^2} \frac{dx}{x \ln x} = \int_a^{a^2} \frac{1}{x \ln x} \, dx = \left[\ln(\ln x)\right]_a^{a^2}$

$= \ln(\ln a^2) - \ln(\ln a) = \ln \frac{\ln a^2}{\ln a} =$

$= \ln \frac{2\cancel{\ln a}}{\cancel{\ln a}} = \ln 2.$

(ιστ) $\int_0^1 \frac{1}{1+e^x} \, dx = \int_0^1 \frac{1+e^x-e^x}{1+e^x} \, dx =$

$= \int_0^1 1 - \frac{e^x}{1+e^x} \, dx = \int_0^1 1 \, dx - \int_0^1 \frac{e^x}{1+e^x} \, dx$

$= (x)_0^1 - \left[\ln(1+e^x)\right]_0^1 = 1 - \left(\ln(1+e) - \ln 2\right).$

(17) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\text{ημ}^2 x\ \text{συν}^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\text{ημ}^2 x + \text{συν}^2 x}{\text{ημ}^2 x\ \text{συν}^2 x}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\text{συν}^2 x} + \frac{1}{\text{ημ}^2 x}\,dx = \left(\text{εφ}x\right)_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \left(\text{σφ}x\right)_{\frac{\pi}{6}}^{\pi/4}$$

$$= 1 - \frac{\sqrt{3}}{3} - \left(1 - \sqrt{3}\right) = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

Ασκηση 5

Να υπολογιστούν τα ολοκληρώματα

(α) $\int_0^1 \frac{2x+3}{x^2+3x+5}\,dx = \left[\ln(x^2+3x+5)\right]_0^1 = \ln 9 - \ln 5$

(β) $\int_2^3 \frac{x+3}{x^2-1}\,dx \overset{(*)}{=}$

$$\frac{x+3}{x^2-1} = \boxed{\frac{x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}}$$

$$x+3 = A(x+1) + B(x-1)$$

$$x+3 = Ax + A + Bx - B$$

$$x+3 = (A+B)x + A - B$$

$$\begin{cases} A+B = 1 \\ A-B = 3 \end{cases} \overset{(+)}{\Rightarrow} 2A = 4 \Leftrightarrow \boxed{A=2} \quad \boxed{B=-1}$$

$(*)$ $\int_2^3 \frac{2}{x-1} - \frac{1}{x+1}\,dx = 2\left[\ln|x-1|\right]_2^3 - \left[\ln|x+1|\right]_2^3$

$$= 2[\ln 2] - [\ln 4 - \ln 3] =$$

$$= 2\ln 2 - \ln 4 + \ln 3 = \ln 3.$$

(γ) $\int_0^1 \frac{2x-3}{x+1}\,dx = \int_0^1 \frac{2x+2-2-3}{x+1}\,dx =$

$= \int_0^1 \frac{2x+2}{x+1} - \frac{5}{x+1}\,dx = \int_0^1 \frac{2(x+1)}{x+1} - \frac{5}{x+1}\,dx$

$= \int_0^1 2 - \frac{5}{x+1}\,dx = 2(x)_0^1 - 5\,\left[\ln|x+1|\right]_0^1 =$

$= 2 - 5\ln 2$

<u>Εναλλακτικά</u>

$\int_0^1 \frac{2x-3}{x+1}\,dx = \int_0^1 \frac{2(x+1)-5}{x+1}\,dx = \int_0^1 2 - \frac{5}{x+1}\,dx$

$$\begin{array}{r|l} 2x-3 & x+1 \\ \hline -(2x+2) & 2 \\ \hline -5 & \end{array}$$

$= 2 - 5\ln 2.$

$\Delta = \delta \cdot \pi + \upsilon$

$\boxed{2x-3 = 2(x+1)-5\,.}$

(δ) $\int_2^3 \frac{x^3+3}{x^2-1}\,dx = \int_2^3 \frac{x(x^2-1)+x+3}{x^2-1}\,dx$

$$\begin{array}{l|l} x^3+3 & x^2-1 \\ \hline -[x^3-x] & x \\ 3+x & \end{array}$$

$\boxed{x^3+3 = x(x^2-1)+x+3}$

$= \int_2^3 x + \frac{x+3}{x^2-1}\,dx =$

$= \int_2^3 x\,dx + \int_2^3 \frac{x+3}{x^2-1}\,dx$

$\stackrel{*}{=}$

$\frac{x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \iff x+3 = A(x+1) + B(x-1)$

$x+3 = Ax + A + Bx - B$

$x+3 = (A+B)x + A - B$

$\begin{cases} 1 = A+B \\ 3 = A-B \end{cases} \stackrel{+}{\Rightarrow} 4 = 2A$

$\boxed{A=2}$

$\boxed{B=-1}$

$\stackrel{*}{=} \frac{1}{2}(x^2)_2^3 + \int_2^3 \frac{2}{x-1} - \frac{1}{x+1}\,dx =$

$= \frac{5}{2} + 2\left[\ln|x-1|\right]_2^3 - \left[\ln|x+1|\right]_2^3 =$

$= \frac{5}{2} + 2[\ln 2] - \ln 4 + \ln 3 = \frac{5}{2} + \ln 3.$

Άσκηση 6

Να υπολογιστούν τα ολοκληρώματα

(α) $\int_0^1 x e^x \, dx = \int_0^1 x (e^x)' \, dx = (x e^x)_0^1 - \int_0^1 e^x (x)' \, dx$

$= e - \int_0^1 e^x \, dx = e - (e^x)_0^1 = e - (e-1)$

$= \underline{1}.$

(β) $\int_0^{\frac{\pi}{2}} x \, ημx \, dx = \int_0^{\pi/2} x (-συνx)' \, dx$

$= (-x \, συνx)_0^{\pi/2} - \int_0^{\pi/2} -συνx \, (x)' \, dx$

$= 0 + \int_0^{\pi/2} συνx \, dx = (ημx)_0^{\pi/2} = \underline{1}.$

(γ) $\int_0^{\pi} x^2 συνx \, dx = \int_0^{\pi} x^2 (ημx)' \, dx =$

$= (x^2 ημx)_0^{\pi} - \int_0^{\pi} ημx \cdot 2x \, dx =$

$= 0 - 2\int_0^{\pi} x \, ημx \, dx = -2\int_0^{\pi} x \, (-συνx)' \, dx =$

$$= -2 \cdot \left[(-x\,\sigma\upsilon\nu x)_0^{\pi} - \int_0^{\pi} -\sigma\upsilon\nu x \, dx \right] =$$

$$= -2 \cdot \left[+\pi - 0 + \int_0^{\pi} \sigma\upsilon\nu x \, dx \right] =$$

$$= -2 \left[\pi + (\eta\mu x)_0^{\pi} \right] = -2\pi .$$

(δ) $\displaystyle\int_1^2 2x \ln x \, dx = \int_1^2 (x^2)' \ln x \, dx =$

$$= \left[x^2 \ln x\right]_1^2 - \int_1^2 x^2 \frac{1}{x} \, dx = 4\ln 2 - \int_1^2 x \, dx$$

$$= 4\ln 2 - \frac{1}{2} (x^2)_1^2 = 4\ln 2 - \frac{3}{2}$$

(ε) $\displaystyle\int_1^2 \ln x \, dx = \int_1^2 1 \cdot \ln x \, dx = \int_1^2 (x)' \ln x \, dx$

$$= (x \ln x)_1^2 - \int_1^2 x \cdot \frac{1}{x} \, dx = 2\ln 2 - \int_1^2 1 \, dx$$

$$= 2\ln 2 - (x)_1^2 = \ln 4 - 1 .$$

(3) $\int_1^2 \frac{\ln x}{x^2}\,dx = \int_1^2 \left(-\frac{1}{x}\right)' \ln x \, dx =$

$= \left(-\frac{1}{x}\ln x\right)_1^2 - \int_1^2 -\frac{1}{x}\,\frac{1}{x}\,dx =$

$= -\frac{1}{2}\ln 2 + \int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{2}\ln 2 - \left(\frac{1}{x}\right)_1^2$

$= -\frac{1}{2}\ln 2 - \left(\frac{1}{2} - 1\right) = \frac{1}{2} - \frac{1}{2}\ln 2$

(4) $\int_1^9 \frac{\ln x}{\sqrt{x}}\,dx = \int_1^9 (2\sqrt{x})' \ln x \, dx =$

$= \left(2\sqrt{x}\ln x\right)_1^9 - \int_1^9 2\sqrt{x}\,\frac{1}{x}\,dx =$

$= 6\ln 9 - 2\int_1^9 \frac{x^{1/2}}{x}\,dx = 6\ln 9 - 2\int_1^9 x^{-\frac{1}{2}}\,dx$

$= 6\ln 2 - 2\left(\frac{x^{1/2}}{\frac{1}{2}}\right)_1^9 = 6\ln 2 - 4\left(\sqrt{x}\right)_1^9$

$= 6\ln 2 - 4 \cdot 2 = 6\ln 2 - 8.$

Ασκηση 7

Να υπολογισετε τα ολοκληρωματα

ⓐ $I = \int_0^{\pi} e^x \sigma\upsilon\nu x \, dx$

$I = \int_0^{\pi} (e^x)' \sigma\upsilon\nu x \, dx = \left(e^x \sigma\upsilon\nu x\right)_0^{\pi} - \int_0^{\pi} e^x (-\eta\mu x) \, dx$

$I = -e^{\pi} - 1 + \int_0^{\pi} e^x \eta\mu x \, dx$

$I = -e^{\pi} - 1 + \int_0^{\pi} (e^x)' \eta\mu x \, dx$

$I = -e^{\pi} - 1 + \left[\left(e^x \eta\mu x\right)_0^{\pi} - \int_0^{\pi} e^x \sigma\upsilon\nu x \, dx \right]$

$I = -e^{\pi} - 1 - I \iff 2I = -e^{\pi} - 1$

$$\boxed{I = -\frac{e^{\pi} + 1}{2}}$$

(B) $$I = \int_0^{\pi} \frac{\eta\mu 2x}{e^{x}}\,dx$$

$$I = \int_0^{\pi} e^{-x}\,\eta\mu 2x\,dx = \int_0^{\pi} (-e^{-x})'\,\eta\mu 2x\,dx$$

$$I = \left(-e^{-x}\,\eta\mu 2x\right)_0^{\pi} - \int_0^{\pi} -e^{-x}\,\sigma\upsilon\nu 2x \cdot 2\,dx$$

$$I = 0 + 2\int_0^{\pi} e^{-x}\,\sigma\upsilon\nu 2x\,dx$$

$$I = 2\int_0^{\pi} (-e^{-x})'\,\sigma\upsilon\nu 2x\,dx$$

$$I = 2\left[\left(-e^{-x}\,\sigma\upsilon\nu 2x\right)_0^{\pi} - \int_0^{\pi} -e^{-x}(-2\eta\mu 2x)\,dx\right]$$

$$I = 2\left[-e^{-\pi} + 1 - 2\int_0^{\pi} e^{-x}\,\eta\mu 2x\,dx\right]$$

$$I = -2e^{-\pi} + 2 - 4I$$

$$5I = -2\left(e^{-\pi}+1\right) \Leftrightarrow I = -2\,\frac{e^{-\pi}+1}{5}$$

$$I = -\frac{2}{5}\cdot\left(e^{-\pi}+1\right)$$

Ασκηση 8

Να υπολογιστούν τα ολοκληρώματα

ⓐ $\int_1^e \frac{\sqrt{\ln x}}{x}\,dx = \int_0^1 \sqrt{t}\,dt = \int_0^1 t^{1/2}\,dt$

$\boxed{\begin{array}{l} \ln x = t \\ \frac{1}{x}dx = 1\cdot dt \\ x=e \Rightarrow t=1 \\ x=1 \Rightarrow t=0 \end{array}}$

$= \left(\frac{t^{3/2}}{3/2}\right)_0^1 = \frac{2}{3}\left(t^{3/2}\right)_0^1 = \frac{2}{3}$

ⓑ $\int_0^1 (x-1)^5(2x-1)\,dx = \int_{-1}^0 t^5\left[2(t+1)-1\right]dt$

$\boxed{\begin{array}{l} x-1=t \\ x=t+1 \\ 1\cdot dx = 1\cdot dt \\ x=1 \quad t=0 \\ x=0 \quad t=-1 \end{array}}$

$= \int_{-1}^0 t^5(2t+2-1)\,dt$

$= \int_{-1}^0 t^5(2t+1)\,dt$

$= \int_{-1}^0 2t^6 + t^5\,dt$

$= 2\left(\frac{t^7}{7}\right)_{-1}^0 + \left(\frac{t^6}{6}\right)_{-1}^0 = \frac{2}{7} - \frac{1}{6}$

(Γ) $\int_0^3 x\sqrt{x+1}\,dx = \int_1^2 (t^2-1)\,t\,2t\,dt =$

$\sqrt{x+1} = t$

$x+1 = t^2$

$x = t^2-1$

$dx = 2t\,dt$

$x=3 \qquad t=2$

$x=0 \qquad t=1$

$= \int_1^2 2t^2(t^2-1)\,dt =$

$= \int_1^2 2t^4 - 2t^2\,dt =$

$= 2\left(\frac{t^5}{5}\right)_1^2 - 2\left(\frac{t^3}{3}\right)_1^2 =$

$= \frac{2}{5}\cdot 31 - \frac{2}{3}\cdot 7$

Ασκηση 9

Να υπολογιστουν τα ολοκληρωματα

ⓐ $\int_0^{\pi} \sigma\upsilon\nu^2 x \; \eta\mu^3 x \, dx = \int_0^{\pi} \sigma\upsilon\nu^2 x \; \eta\mu^2 x \; \eta\mu x \, dx =$

$= \int_0^{\pi} \sigma\upsilon\nu^2 x (1 - \sigma\upsilon\nu^2 x) \; \eta\mu x \, dx = -\int_1^{-1} t^2(1-t^2)\, dt$

Θετω $\sigma\upsilon\nu x = t$

$-\eta\mu x \, dx = dt$

$x = \pi \Leftrightarrow t = -1$

$x = 0 \Leftrightarrow t = 1$

$= \int_{-1}^{1} t^2 - t^4 \, dt$

$= \frac{1}{3}(t^3)_{-1}^{1} - \frac{1}{5}(t^5)_{-1}^{1}$

$= \frac{2}{3} - \frac{2}{5} = \frac{10}{15} - \frac{6}{15} = \frac{4}{15}$

ⓑ $\int_0^{\frac{\pi}{2}} \sigma\upsilon\nu^3 x \, dx = \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu x \; \sigma\upsilon\nu^2 x \, dx = \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu x (1 - \eta\mu^2 x) \, dx$

Θετω $\eta\mu x = t$

$\sigma\upsilon\nu x \, dx = dt$

$x = \frac{\pi}{2} \Leftrightarrow t = 1$

$x = 0 \Leftrightarrow t = 0$

$= \int_0^1 1 - t^2 \, dt =$

$= (t)_0^1 - \frac{1}{3}(t^3)_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

(γ) $\int_0^{\frac{\pi}{2}} \eta\mu^4 x\,dx = \int_0^{\frac{\pi}{2}} \eta\mu^2 x\,\eta\mu^2 x\,dx =$

$= \int_0^{\frac{\pi}{2}} \frac{1-\sigma\upsilon\nu 2x}{2}\cdot\frac{1-\sigma\upsilon\nu 2x}{2} = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\sigma\upsilon\nu 2x)^2\,dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 1 - 2\sigma\upsilon\nu 2x + \sigma\upsilon\nu^2 2x\,dx =$

$= \frac{1}{4}\left[\int_0^{\pi/2} 1\,dx - 2\int_0^{\pi/2} \sigma\upsilon\nu 2x\,dx + \int_0^{\pi/2} \sigma\upsilon\nu^2 2x\,dx\right]$

$= \frac{1}{4}\left[(x)_0^{\pi/2} - 2\,\frac{1}{2}(\eta\mu 2x)_0^{\pi/2} + \int_0^{\pi/2} \frac{1+\sigma\upsilon\nu 2x}{2}\,dx\right]$

$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{1}{2}\int_0^{\pi/2} 1+\sigma\upsilon\nu 2x\,dx\right] =$

$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{1}{2}(x)_0^{\pi/2} + \frac{1}{2}\,\frac{1}{2}(\eta\mu 2x)_0^{\pi/2}\right]$

$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{1}{2}\,\frac{\pi}{2}\right] = \frac{1}{4}\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$

Ασκηση 12

Εστω f συνεχης στο $[-\alpha, \alpha]$

ⓐ Αν η f περιττη τοτε $\int_{-\alpha}^{\alpha} f(x)\, dx = 0$

$$I = \int_{-a}^{a} f(x)\, dx = \int_{\alpha}^{-\alpha} -f(-t)\, dt = \int_{-\alpha}^{a} -f(t)\, dt =$$

$$\boxed{\begin{aligned} &\text{Θετω } x = -t \\ &dx = -dt \end{aligned}}$$

$$= -\int_{-a}^{a} f(t)\,dt = -I.$$

Δηλαδη $I = -I \Leftrightarrow 2I = 0 \Leftrightarrow I = 0$

ⓑ Αν η f ειναι αρτια τοτε $\int_{-\alpha}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

$$\int_{-\alpha}^{a} f(x)\, dx = \int_{-\alpha}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

• $\int_{-\alpha}^{0} f(x)\, dx = \int_{\alpha}^{0} -f(-t)\, dt = \int_{0}^{\alpha} f(t)\, dt$

$$\boxed{\begin{aligned} &\text{Θετω } x = -t \\ &dx = -dt \end{aligned}}$$

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$\int_{-\alpha}^{a} f(x)\, dx = 2\int_{0}^{\alpha} f(x)\, dx$$

Ασκηση: 14

Δινεται $f(x) = e^x + x^3$

Να υπολογιστει το $\int_1^{e+1} f^{-1}(x)\,dx$ εφοσον f^{-1} παρ/μη

$$\int_1^{e+1} f^{-1}(x)\,dx = \int_0^1 t\,f'(t)\,dt = \int_0^1 t(e^t+3t^2)\,dt$$

$$\boxed{\begin{aligned} \text{Θετω } f^{-1}(x) &= t \\ x &= f(t) \\ dx &= f'(t)\,dt \end{aligned}}$$

$$= \int_0^1 te^t + 3t^3\,dt =$$

$$= \int_0^1 te^t\,dt + 3\int_0^1 t^3\,dt =$$

$$= \left(te^t\right)_0^1 - \int_0^1 e^t\,dt + 3\,\frac{1}{4}\left(t^4\right)_0^1 =$$

$$= e - \left(e^t\right)_0^1 + \frac{3}{4} = e - (e-1) + \frac{3}{4} =$$

$$= 1 + \frac{3}{4} = \frac{7}{4}$$

Ασκηση 11

Να υπολογιστει το ολοκληρωμα

$$I = \int_{-1}^{1} \frac{x^2}{e^x+1}\,dx = \int_{1}^{-1} -\frac{t^2}{e^{-t}+1}\,dt = \int_{-1}^{1} \frac{t^2}{\frac{1}{e^t}+1}\,dt$$

Θετω $t=-x$

$\Leftrightarrow x=-t$

$dx=-dt$

$$= \int_{-1}^{1} \frac{t^2}{\frac{1+e^t}{e^t}}\,dt =$$

$$= \int_{-1}^{1} \frac{t^2 e^t}{e^t+1}\,dt$$

Συνεπως $I = \int_{-1}^{1} \frac{x^2}{e^x+1}\,dx = \int_{-1}^{1} \frac{t^2 e^t}{e^t+1}\,dt$

$$\Leftrightarrow I + I = \int_{-1}^{1} \frac{x^2}{e^x+1}\,dx + \int_{-1}^{1} \frac{t^2 e^t}{e^t+1}\,dt$$

$$2I = \int_{-1}^{1} \frac{x^2+x^2e^x}{e^x+1}\,dx = \int_{-1}^{1} \frac{x^2(e^x+1)}{e^x+1}\,dx$$

$$2I = \int_{-1}^{1} x^2\,dx \Leftrightarrow 2I = \frac{1}{3}\left(x^3\right)_{-1}^{1}$$

$$2I = \frac{2}{3} \Leftrightarrow I = \frac{1}{3}$$