Θεματικη Ενοτητα

Υπολογισμος

ολοκληρωματων

1. $\int_0^1 x^3-2x^2+1 \; dx =$

$$= \int_0^1 x^3\,dx - \int_0^1 2x^2\,dx + \int_0^1 1\,dx =$$

$$= \frac{1}{4}\left(x^4\right)_0^1 - \frac{2}{3}\left(x^3\right)_0^1 + \left(x\right)_0^1 =$$

$$= \frac{1}{4} - \frac{2}{3} + 1 \quad = \frac{3}{12} - \frac{8}{12} + \frac{12}{12} = \frac{7}{12}$$

2. $\int_1^2 \frac{(x-1)^2}{x}\,dx = \int_1^2 \frac{x^2-2x+1}{x}\,dx$

$$= \int_1^2 \frac{x^2}{x} - \frac{2x}{x} + \frac{1}{x}\,dx = \int_1^2 x + 2 + \frac{1}{x}\,dx =$$

$$= \int_1^2 x\,dx + \int_1^2 2\,dx + \int_1^2 \frac{1}{x}\,dx$$

$$= \frac{1}{2}\left(x^2\right)_1^2 + 2\left(x\right)_1^2 + \left(\ln x\right)_1^2$$

$$= \frac{3}{2} + 2 + \ln 2$$

3. Παραγοντική ολοκλήρωση

(α) $\int_0^{\pi} x\eta\mu x \, dx = \int_0^{\pi} x(-\sigma\upsilon\nu x)' \, dx =$

$= (-x\sigma\upsilon\nu x)_0^{\pi} - \int_0^{\pi} -\sigma\upsilon\nu x \, dx =$

$= -(\pi\sigma\upsilon\nu\pi - 0) + \int_0^{\pi} \sigma\upsilon\nu x \, dx =$

$= \pi + (\eta\mu x)_0^{\pi} = \pi$

Σχόλιο

$\int P(x)\, e^x \, dx$ $\qquad$ $\int P(x)\, \eta\mu x \, dx$

$\int P(x)\, \sigma\upsilon\nu x \, dx$ $\qquad$ $\int P(x)\, \ln x \, dx$

(β) $\int_1^{e} \ln x \, dx = \int_1^{e} 1\cdot \ln x \, dx = \int_1^{e} (x)' \ln x \, dx$

$= (x\ln x)_1^{e} - \int_1^{e} x \cdot \frac{1}{x} \, dx =$

$= e - \int_1^{e} 1 \, dx = e - (x)_1^{e} = e - (e-1)$

$= 1.$

4. $$I=\int_0^{\pi} e^x \,\eta\mu x \,dx$$

$$I=\int_0^{\pi} (e^x)' \,\eta\mu x \,dx$$

$$I=\left(e^x \,\eta\mu x\right)_0^{\pi} - \int_0^{\pi} e^x \,\sigma\upsilon\nu x \,dx$$

$$I=-\int_0^{\pi} (e^x)' \,\sigma\upsilon\nu x \,dx$$

$$I=-\left(e^x \,\sigma\upsilon\nu x\right)_0^{\pi} + \int_0^{\pi} e^x(-\eta\mu x)\,dx$$

$$I=-\left(e^{\pi}\,\sigma\upsilon\nu \pi - 1\right) - \int_0^{\pi} e^x \,\eta\mu x \,dx$$

$$I=+e^{\pi}+1-I$$

$$2I=e^{\pi}+1$$

$$I=\frac{e^{\pi}+1}{2}.$$

5. $\int_0^1 e^{2x+1}\,dx = \left(\frac{e^{2x+1}}{2}\right)_0^1 =$

$= \frac{1}{2}\left(e^{2x+1}\right)_0^1 = \frac{1}{2}\left(e^3 - e\right)$

<u>Σχολιο</u>

$\int \eta\mu(\alpha x+\beta)\,dx = \frac{-\sigma\upsilon\nu(\alpha x+\beta)}{\alpha}$

$\int \frac{1}{\alpha x+\beta}\,dx = \frac{\ln|\alpha x+\beta|}{\alpha}$

$\int e^{\alpha x+\beta}\,dx = \frac{e^{\alpha x+\beta}}{\alpha}$

<u>Προσοχη</u>

$\int_0^1 2^x\,dx = \left(\frac{2^x}{\ln 2}\right)_0^1 = \frac{1}{\ln 2}$

Δηλαδη $\int \alpha^x\,dx = \frac{\alpha^x}{\ln\alpha}$

6. Ρητά ολοκληρώματα

(α) $\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx =$

$= \frac{1}{2}\left(\ln|x^2+1|\right)_0^1 = \frac{1}{2}\ln 2$

(β) $\int_0^2 \frac{x+2}{x^2-1}\,dx = \int_0^2 \frac{\frac{3}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx$

$= \frac{3}{2}\left(\ln|x-1|\right)_0^2 - \frac{1}{2}\left(\ln|x+1|\right)_0^2$

$= -\frac{1}{2}\ln 3$

$\boxed{\text{Μέθοδος } A, B}$

$\frac{x+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$

$x+2 = A(x+1) + B(x-1)$

$x+2 = Ax + A + Bx - B$

$x+2 = (A+B)\,x + A - B$

$\begin{cases} A+B = 1 \\ A-B = 2 \end{cases} \stackrel{(+)}{} \quad 2A = 3 \Rightarrow A = \frac{3}{2}$

$\frac{3}{2} + B = 1 \Rightarrow B = -\frac{1}{2}$

γ) $$\int_0^1 \frac{x^3}{x^2+1}\,dx = \int_0^1 \frac{(x^2+1)x - x}{x^2+1}\,dx =$$

$$\begin{array}{r|l} x^3 & x^2+1 \\ \hline -(x^3+x) & x \\ \hline -x & \end{array}$$

$$= \int_0^1 \frac{\cancel{(x^2+1)}\,x}{\cancel{x^2+1}} - \frac{x}{x^2+1}\,dx$$

$$= \int_0^1 x\,dx - \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= \frac{1}{2}\left(x^2\right)_0^1 - \frac{1}{2}\left(\ln|x^2+1|\right)_0^1 =$$

$$= \frac{1}{2} - \frac{1}{2}\ln 2$$

7. ⓐ $\int_1^4 \sqrt{x}\,dx = \int_1^2 t\,2t\,dt =$

$\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

$= 2\int_1^2 t^2\,dt = \frac{2}{3}\left(t^3\right)_1^2$

$= \frac{2}{3}\,7 = \frac{14}{3}$

ⓑ $\int_2^5 x\sqrt{x-1}\,dx = \int_1^2 (t^2+1)\,t\,2t\,dt$

$\sqrt{x-1} = t$

$x-1 = t^2$

$x = t^2+1$

$dx = 2t\,dt$

$= 2\int_1^2 t^2(t^2+1)\,dt$

$= 2\int_1^2 t^4 + t^2\,dt$

$= \frac{2}{5}\left(t^5\right)_1^2 + \frac{2}{3}\left(t^3\right)_1^2$

$= \frac{62}{5} + \frac{14}{3}$

(γ) $\int_1^e \frac{\ln x+1}{x}\,dx = \int_0^1 t+1\,dt =$

$\ln x = t$

$\frac{1}{x}\,dx = dt$

$= \frac{1}{2}\left(t^2\right)_0^1 + \left(t\right)_0^1 =$

$= \frac{1}{2} + 1 = \frac{3}{2}$

8. Τεχνάσματα

(α). $\int_0^{\frac{\pi}{3}} εφx\, dx = \int_0^{\frac{\pi}{3}} \frac{ημx}{συνx}\, dx =$

$= -\left(\ln|συνx|\right)_0^{\pi/3} = -\left(\ln\frac{1}{2} - \ln 1\right) = -\ln\frac{1}{2}$

(β) $\int_0^1 \frac{1}{1+e^x}\, dx = \int_0^1 \frac{1+e^x-e^x}{1+e^x}\, dx$

$= \int_0^1 \cancel{\frac{1+e^x}{1+e^x}} - \int_0^1 \frac{e^x}{1+e^x}\, dx$

$= \int_0^1 1\, dx - \left(\ln|1+e^x|\right)_0^1 =$

$= (x)_0^1 - \left(\ln(1+e) - \ln 2\right)$

$= 1 - \ln(1+e) + \ln 2$

(γ) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\eta\mu^2 x\,\sigma\upsilon\nu^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\eta\mu^2 x + \sigma\upsilon\nu^2 x}{\eta\mu^2 x\,\sigma\upsilon\nu^2 x}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cancel{\eta\mu^2 x}}{\cancel{\eta\mu^2 x}\,\sigma\upsilon\nu^2 x}\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cancel{\sigma\upsilon\nu^2 x}}{\eta\mu^2 x\,\cancel{\sigma\upsilon\nu^2 x}}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sigma\upsilon\nu^2 x}\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\eta\mu^2 x}\,dx$$

$$= (\varepsilon\varphi x)_{\frac{\pi}{6}}^{\frac{\pi}{4}} - (\sigma\varphi x)_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$$

$$= 1 - \frac{\sqrt{3}}{3} - \left(1 - \sqrt{3}\right) = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

(δ) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x}{\text{συν}^2 x}\,dx = \int_0^{\frac{\pi}{4}} x\,\frac{1}{\text{συν}^2 x}\,dx$

$\displaystyle= \int_0^{\frac{\pi}{4}} x\,(\text{εφ}x)'\,dx =$

$\displaystyle= \left(x\,\text{εφ}x\right)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \text{εφ}x\,dx =$

$\displaystyle= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\text{ημ}x}{\text{συν}x}\,dx =$

$\displaystyle= \frac{\pi}{4} + \left(\ln|\text{συν}x|\right)_0^{\pi/4} = \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2} - \ln 1$

(ε) $\displaystyle\int_0^{\pi} \text{ημ}^3 x\,dx = \int_0^{\pi} \text{ημ}x\,\text{ημ}^2 x\,dx =$

$\displaystyle= \int_0^{\pi} \text{ημ}x\,(1-\text{συν}^2 x)\,dx \quad \frac{\text{συν}x = t}{-\text{ημ}x\,dx = dt}$

$\displaystyle= -\int_1^{-1} 1-t^2\,dt = \int_1^{-1} t^2-1\,dt =$

$\displaystyle= \frac{1}{3}\left(t^3\right)_1^{-1} - \left(t\right)_1^{-1} = -\frac{2}{3} + 2 = \frac{4}{3}.$

(ϑ) $\int_0^{\pi} \eta\mu^2 x \, dx \overset{(*)}{=}$

Τυπου αποτετραγωνισμου

$\sigma\upsilon\nu^2 x = \dfrac{1+\sigma\upsilon\nu 2x}{2}$

$\eta\mu^2 x = \dfrac{1-\sigma\upsilon\nu 2x}{2}$

$\overset{(*)}{=} \int_0^{\pi} \dfrac{1-\sigma\upsilon\nu 2x}{2} dx = \int_0^{\pi} \frac{1}{2} dx - \int_0^{\pi} \dfrac{\sigma\upsilon\nu 2x}{2} dx$

$= \frac{1}{2} (x)_0^{\pi} - \frac{1}{2} \left(\dfrac{\eta\mu 2x}{2}\right)_0^{\pi} =$

$= \frac{1}{2}\pi - \frac{1}{4} \cdot 0 = \frac{\pi}{2}.$

(η) $\int_0^{\pi} \eta\mu^3 x \, \sigma\upsilon\nu^2 x \, dx = \int_0^{\pi} \eta\mu x \, \eta\mu^2 x \, \sigma\upsilon\nu^2 x \, dx$

$= \int_0^{\pi} \eta\mu x \, (1-\sigma\upsilon\nu^2 x) \, \sigma\upsilon\nu^2 x \, dx \; \underset{-\eta\mu x \, dx = dt}{\overset{\sigma\upsilon\nu x = t}{=\!=\!=}}$

$= -\int_1^{-1} (1-t^2) t^2 \, dt = \int_1^{-1} (t^2-1) t^2 \, dt =$

$= \int_1^{-1} t^4 - t^2 dt = \frac{1}{5}(t^5)_1^{-1} - \frac{1}{3}(t^3)_1^{-1} = -\frac{2}{5} + \frac{2}{3}.$

θ) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\text{συν}\, x}\,dx = \int_0^{\pi/6} \frac{\text{συν}\, x}{\text{συν}^2 x}\,dx =$

$$= \int_0^{\pi/6} \frac{\text{συν}\, x}{1-\text{ημ}^2 x}\,dx \;\underset{\text{συν}\,x\,dx = dt}{\overset{\text{ημ}\,x = t}{=\!=\!=\!=}}$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1-t^2}\,dt \;\overset{(*)}{=\!=\!=}$$

$$\frac{1}{(1-t)(1+t)} = \frac{A}{1-t} + \frac{B}{1+t} \iff 1 = A(1+t) + B(1-t)$$

$$1 = A + At + B - Bt$$

$$1 = (A-B)\,t + A + B$$

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases} \overset{(+)}{\iff} 2A = 1 \implies A = \frac{1}{2}, \quad B = \frac{1}{2}$$

$$\overset{(*)}{=\!=\!=} \int_0^{1/2} \frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t}\,dt = \frac{1}{2}\Big(\ln|1-t|\Big)_0^{1/2} + \frac{1}{2}\Big(\ln|1+t|\Big)_0^{1/2}$$

$$= \frac{1}{2}\ln\frac{1}{2} + \frac{1}{2}\ln\frac{3}{2}$$

9. Αρτια και περιττη

ⓐ Εστω περιττη συναρτηση $f(x)$

Τοτε $\int_{-a}^{a} f(x) = 0$

Αποδειξη

$$I = \int_{-a}^{a} f(x)\,dx \xlongequal[dx=-dt]{x=-t} -\int_{a}^{-a} f(-t)\,dt$$

$$I = \int_{-a}^{a} -f(t)\,dt = -\int_{-a}^{a} f(t)\,dt$$

$$I = -I \Leftrightarrow 2I = 0 \Rightarrow I = 0.$$

π.χ

$$\int_{-2}^{2} \underbrace{\frac{e^x-1}{e^x+1}}_{f(x)}\,dx = 0.$$

$$f(-x) = \frac{e^{-x}-1}{e^{-x}+1} = \frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1} = \frac{1-e^x}{1+e^x} = -\frac{e^x-1}{e^x+1} = -f(x)$$

περιττη.

(B) Εστω f ειναι αρτια.

Τοτε $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

απoδειξη

$$I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\rightarrow \int_{-a}^{0} f(x)\,dx \;\underset{dx=-dt}{\overset{x=-t}{=\!=}}\; -\int_{a}^{0} f(-t)\,dt =$$

$$= \int_{0}^{a} f(t)\,dt$$

Αρα $I = \int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx$

$$I = 2\int_{0}^{a} f(x)\,dx.$$

(γ) Έστω ότι $f(x) = 2 - f(2-x)$

$$I = \int_0^2 f(x)\,dx \xrightarrow[dx=-dt]{x=2-t} -\int_2^0 f(2-t)\,dt$$

$$= \int_0^2 2 - f(t)\,dt = \int_0^2 2\,dt - \int_0^2 f(t)\,dt$$

$$I = 2(x)_0^2 - I$$

$$2I = 4$$

$$\underline{I = 2}$$