Ευρεση Τυπου

1. $f: R \to R$ ωστε $f(x+1) = (x+1)e^{-x}$

Ψαχνω τυπο $f(x)$

Θετω $x+1 = t$

$x = t-1$ αρα $f(t) = t\, e^{-(t-1)}$

$f(t) = t\, e^{1-t}$

η $\boxed{f(x) = x e^{1-x}}$

2. $f: (0,+\infty) \to R$

- $f\left(\frac{x}{e}\right) \le \ln x \, , x>0$
- $-f\left(\frac{1}{x}\right) + 1 \le \ln x , x>0$

Θετω $\frac{x}{e} = t \Rightarrow x = te$

$f(t) \le \ln(te)$

$f(t) \le \ln t + \ln e$

$f(t) \le \ln t + 1$

$f(x) \le \ln x + 1$

Θετω $\frac{1}{x} = t$

$x = \frac{1}{t}$

$-f(t) + 1 \le \ln \frac{1}{t}$

$-f(t) + 1 \le \ln 1 - \ln t$

$-f(t) + 1 \le -\ln t$

$f(t) - 1 \ge \ln t$

$f(x) \ge \ln x + 1$

$\boxed{f(x) = \ln x + 1}$

3. $f: \mathbb{R} \to \mathbb{R}$ συνεχής. $\quad x^2 f\left(\frac{1}{x}\right) = ημx \quad \forall x \neq 0.$

Ψάχνω τύπο $f(x)$

Θέτω $\frac{1}{x} = t \Rightarrow x = \frac{1}{t}$

$$\left(\frac{1}{t}\right)^2 f(t) = ημ\frac{1}{t}$$

$$\frac{1}{t^2} f(t) = ημ\frac{1}{t} \Rightarrow f(t) = t^2 ημ\frac{1}{t}$$

$$\boxed{f(x) = x^2 ημ\frac{1}{x} \qquad x \neq 0}$$

$$f(0) \overset{\text{συνεχής}}{=\!=} \lim_{x\to 0} x^2 ημ\frac{1}{x} = 0 \Rightarrow \underline{\underline{f(0) = 0}}$$

$$-1 \leq ημ\frac{1}{x} \leq 1$$

$$\boxed{-x^2 \leq x^2 ημ\frac{1}{x} \leq x^2}$$

$$\left.\begin{array}{l} \lim_{x\to 0} -x^2 = 0 \\ \lim_{x\to 0} x^2 = 0 \end{array}\right\} \text{Από Κ.Π.} \lim_{x\to 0} x^2 ημ\frac{1}{x} = 0$$

$$f(x) = \begin{cases} x^2 ημ\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

4. $f:\mathbb{R} \to \mathbb{R}$ συνεχής $f(0)=1$.

$$e^{2x} f^2(x) - 2x^2 e^x f(x) = 2x^2+1 \quad \forall x \in \mathbb{R}.$$

Ψάχνω τύπο $f(x)$

$$e^{2x} f^2(x) - 2x^2 e^x f(x) + x^4 = x^4 + 2x^2 + 1$$

$$\left(e^x f(x) - x^2\right)^2 = \left(x^2+1\right)^2$$

$$|\underbrace{e^x f(x) - x^2}_{h(x)}| = |x^2+1|$$

$$|h(x)| = x^2+1$$

Ρίζα $h(x)$

$h(x) = 0$

$|h(x)| = 0$

$x^2+1 = 0$

Άτοπο

$h(x) \neq 0$ και συνεχής

$h(x) > 0$ ή $h(x) < 0$

$h(0) = e^0 f(0) - 0^2 = 1$

$h(x) > 0$

$h(x) = x^2+1$

$e^x f(x) - x^2 = x^2+1$

$e^x f(x) = 2x^2+1$

$$\boxed{f(x) = \frac{2x^2+1}{e^x}}$$

5. Έστω $f: [-1, 4] \to R$ συνεχής.

και $x^2 + f^2(x) = 3x+4 \quad \forall x \in [-1, 4]$

$f(0) = -2$ Ψάχνω τύπο $f(x)$

$$f^2(x) = -x^2 + 3x + 4$$

x		-1		4	
$-x^2+3x+4$	$-$		$+$		$-$

$$f^2(x) = \sqrt{-x^2+3x+4}^{\,2}$$

$$|f(x)| = \left|\sqrt{-x^2+3x+4}\right|$$

+

−

$$|f(x)| = \sqrt{-x^2+3x+4}$$

$$-f(x) = \sqrt{-x^2+3x+4}$$

$$\boxed{f(x) = -\sqrt{-x^2+3x+4}}$$

Ρίζα $f(x)$

$f(x) = 0$

$|f(x)| = 0$

$\sqrt{-x^2+3x+4} = 0$

$-x^2+3x+4 = 0$

$x = -1$ $\quad x = 4$

x		-1		4	
$f(x)$		0	$-$	0	

$f(0) = -2$

6. Να βρω όλ τις συνεχ συναρτησ.

$f: \mathbb{R} \to \mathbb{R}$ ώστε $f^2(x) = \left(e^{x^2} - x^2 - 1\right)^2$

$$f^2(x) = \left(e^{x^2} - x^2 - 1\right)^2$$

$$|f(x)| = \left|e^{x^2} - x^2 - 1\right| \quad (+)$$

$$|f(x)| = e^{x^2} - x^2 - 1$$

$e^x \geq x + 1$

$e^{x^2} \geq x^2 + 1$

$e^{x^2} - x^2 - 1 \geq 0$

Ρίζα $f(x)$

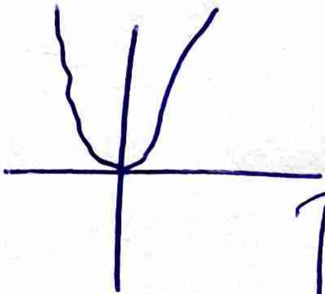
$f(x) = 0$

$|f(x)| = 0$

$e^{x^2} - x^2 - 1 = 0$

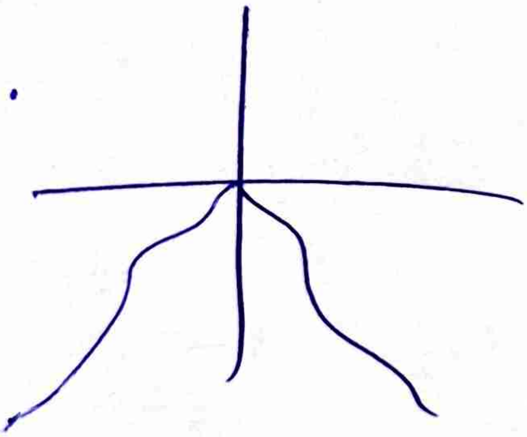
$x^2 = 0$

$\boxed{x = 0}$

1. 

$(+)$ $|f(x)| = e^{x^2} - x^2 - 1$

$\boxed{f(x) = e^{x^2} - x^2 - 1}$

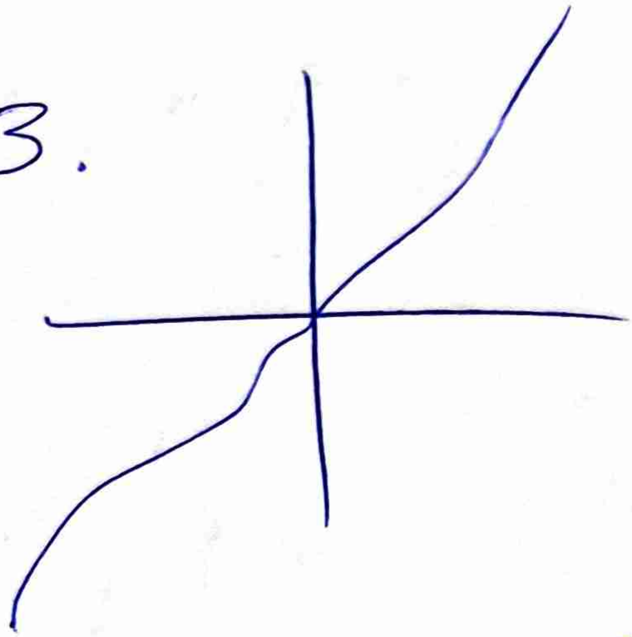
2. 

$(-)$ $|f(x)| = e^{x^2} - x^2 - 1$

$-f(x) = e^{x^2} - x^2 - 1$

$\boxed{f(x) = -e^{x^2} + x^2 + 1}$

3.



$$|f(x)| = e^{x^2} - x^2 - 1$$

$\underline{x<0}$

$$-f(x) = e^{x^2} - x^2 - 1$$

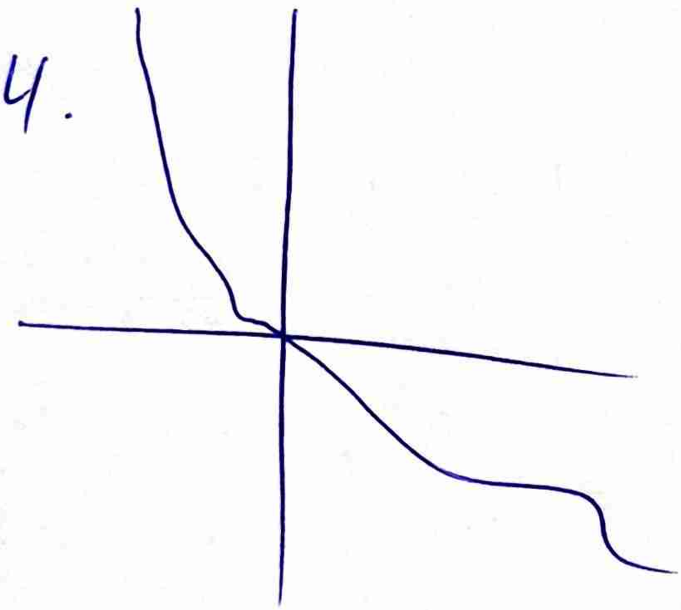
$$f(x) = -e^{x^2} + x^2 + 1$$

$\underline{x>0}$

$$f(x) = e^{x^2} - x^2 - 1$$

$$f(x) = \begin{cases} -e^{x^2} + x^2 + 1, & x \le 0 \\ e^{x^2} - x^2 - 1, & x > 0 \end{cases}$$

4.



$$f(x) = \begin{cases} e^{x^2} - x^2 - 1, & x \le 0 \\ -e^{x^2} + x^2 + 1, & x > 0. \end{cases}$$

7. Έστω $f:(0,+\infty)\rightarrow R$ παραγωγίσιμη.

$f(4)=3$ και $2x\,f'(x)+f(x)=0 \quad \forall x>0$

Νδο $g(x)=f(x)\sqrt{x}$ είναι σταθερή

και στη συνέχεια να βρεί τύπο $f(x)$

Λύση

$$\rightarrow \quad g'(x)=f'(x)\sqrt{x}+f(x)\frac{1}{2\sqrt{x}}$$

$$g'(x)=\frac{2\sqrt{x}\,f'(x)\sqrt{x}}{2\sqrt{x}}+\frac{f(x)}{2\sqrt{x}}$$

$$g'(x)=\frac{2x\,f'(x)+f(x)}{2\sqrt{x}}=0$$

$g'(x)=0$ άρα $g(x)=c$

άρα $f(x)\sqrt{x}=c$

$$\underline{x=4}$$

$$f(4)\sqrt{4}=c$$

$$3\cdot 2=c$$

$$c=6$$

$$f(x)=\frac{6}{\sqrt{x}}$$

8. Δίνεται $f: \mathbb{R}^* \to \mathbb{R}$ και $f(2)=2$

και $f(-1)=-3$ του $f'(x) = -\dfrac{2f(x)}{x}, x\neq 0$

Νδο η $g(x) = x^2 f(x)$ σταθερη και να

βρεθεί τυπο $f(x)$

<u>Λυση</u>

$$g'(x) = 2xf(x) + x^2 f'(x) = 2xf(x) + x^2\left(-\frac{2f(x)}{x}\right)$$

$$g'(x) = 2xf(x) - 2xf(x) = 0$$

$$g'(x) = 0$$

$g'(x)=0$ | $g'(x)=0$ (στο 0)

$$g(x) = \begin{cases} c_1, & x<0 \\ c_2, & x>0 \end{cases}$$

$$x^2 f(x) = \begin{cases} c_1, & x<0 \\ c_2, & x>0 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{c_1}{x^2}, & x<0 \\ \dfrac{c_2}{x^2}, & x>0 \end{cases}$$

$f(-1) = \dfrac{c_1}{1} = -3$, $c_1 = -3$

$f(2) = \dfrac{c_2}{4} = 2 \Rightarrow c_2 = 8$

$$f(x) = \begin{cases} \dfrac{-3}{x^2}, & x<0 \\ \dfrac{8}{x^2}, & x>0 \end{cases}$$

9. $f: \mathbb{R} \to \mathbb{R}$ και $f(0) = 0$ παρ/μη $(-\infty,-1) \cup (-1,+\infty)$

$$f'(x) = \begin{cases} -2 & , x < -1 \\ 3x^2 - 1 & , x > -1 \end{cases}$$

Βρε τύπο $f(x)$

$\underline{x < -1}$

$f'(x) = -2$

$f'(x) = (-2x)'$

$f(x) = -2x + C$

$\underline{x > -1}$

$f'(x) = 3x^2 - 1$

$f'(x) = (x^3 - x)'$

$f(x) = x^3 - x + C$

$\underline{x = 0}$

$f(0) = C$

$\underline{\underline{C = 0}}$

$$f(x) = \begin{cases} -2x + C, & x < -1 \\ x^3 - x, & x \geq -1 \end{cases}$$

$$\left.\begin{aligned} \lim_{x \to -1^-} f(x) &= 2 + C \\ \lim_{x \to -1^+} f(x) &= 0 \end{aligned}\right\} \begin{aligned} 2 + C &= 0 \\ C &= -2 \end{aligned}$$

$$f(x) = \begin{cases} -2x - 2, & x < -1 \\ x^3 - x, & x \geq -1 \end{cases}$$

10. Έστω $f:[0,+\infty) \to R$ και

$$f'(x) = \frac{1}{2\sqrt{x}} + 3 \quad \forall x>0 \quad \text{και} \quad f(4)=15$$

Βρείτε την $f(x)$

<u>Λύση</u>

$$f'(x) = (\sqrt{x} + 3x)'$$

$$f(x) = \sqrt{x} + 3x + c$$

<u>$x=4$</u>

$$f(4) = 2 + 12 + c$$

$$15 = 14 + c$$

$$c = 1$$

$$\boxed{f(x) = \sqrt{x} + 3x + 1}$$

11. $f : [0, +\infty) \longrightarrow \mathbb{R}$ συνεχής.

$f(x)\, f'(x) = \frac{1}{2}, \ x > 0$ $\qquad f(1) = 1$

Βρεί τύπο $f(x)$

<u>Λύση</u>

$2 f(x) f'(x) = 1$

$(f^2(x))' = (x)'$

$f^2(x) = x + c$

$\underline{x = 1}$

$f^2(1) = 1 + c$

$1 = 1 + c$

$c = 0$

$f^2(x) = x$

$f^2(x) = \sqrt{x}^{\,2}$

$+$

$|f(x)| = |\sqrt{x}|$

$|f(x)| = \sqrt{x}$

P.Td $f(x)$

$f(x) = 0$

$|f(x)| = 0$

$\sqrt{x} = 0$

$x = 0$

x		0	
$f(x)$		0	+

$f(1) = 1$

$|\overset{+}{f(x)}| = \sqrt{x}$

$f(x) = \sqrt{x}$

SOS

$$(f^2(x))' = 2f(x)f'(x)$$

$$\left(e^{f(x)}\right)' = f'(x)e^{f(x)}$$

$$(\ln f(x))' = \frac{f'(x)}{f(x)}$$

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}$$

12. Έστω $f: \mathbb{R} \to \mathbb{R}$ παραγωγίσιμη

και $(x-1) f'(x) = 2x^2 + 3x - 5 \quad \forall x \in \mathbb{R}$. $f(2) = 11$

Βρεί τύπο $f(x)$

$$f'(x) = \frac{(2x+5)\cancel{(x-1)}}{\cancel{x-1}}, \quad x \neq 1$$

$$f'(x) = 2x + 5$$

$$f'(x) = (x^2 + 5x)'$$

$$f(x) = x^2 + 5x + c \;; \quad \text{ΟΧΙ!}$$

Είναι σπάω το διάστημα.

$$f(x) = \begin{cases} x^2 + 5x + c_1, & x < 1 \\ \\ x^2 + 5x + c_2, & x \geq 1 \end{cases}$$

$$f(2) = 4 + 10 + c_2$$

$$11 = 14 + c_2$$

$$c_2 = -3$$

$$\Rightarrow \underline{\underline{f(x) = x^2 + 5x - 3}}$$

Άρα f συνεχής

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 5x + c_1 = 6 + c_1$$

$$\lim_{x \to 1^+} f(x) = 6 - 3 = 3$$

$$\Leftrightarrow 6 + c_1 = 3$$

$$c_1 = -3.$$

13. $f: \mathbb{R} \to \mathbb{R}$ παραγωγίσιμη $\quad f(1) = \ln 3$

$$f'(x) = 4x^3 e^{-f(x)}$$

Βρεί τύπο $f(x)$

$$f'(x) = 4x^3 \frac{1}{e^{f(x)}}$$

$$f'(x) e^{f(x)} = 4x^3$$

$$\left(e^{f(x)}\right)' = \left(x^4\right)'$$

$$e^{f(x)} = x^4 + c$$

$\underline{x=1}$

$$e^{f(1)} = 1 + c$$

$$e^{\ln 3} = 1 + c$$

$$3 = 1 + c$$

$$c = 2$$

$$e^{f(x)} = x^4 + 2$$

$$\underline{f(x) = \ln(x^4 + 2)}$$

14. Έστω $f: \mathbb{R} \to \mathbb{R}$ παρ/μη $\underline{\underline{f(x) > 0}}$

$\forall x \in \mathbb{R}$ και $f(1) = e^2$ και $f'(x) = 2x f(x)$

$\forall x \in \mathbb{R}$.

Βρείτε τον $f(x)$

$$f'(x) = 2x f(x)$$

$$\frac{f'(x)}{f(x)} = 2x$$

$$\left(\ln(f(x)) \right)' = (x^2)'$$

$$\ln f(x) = x^2 + c$$

$\underline{x = 1}$

$$\ln f(1) = 1 + c$$

$$\ln e^2 = 1 + c$$

$$2 = 1 + c$$

$$c = 1$$

$$\ln f(x) = x^2 + 1$$

$$f(x) = e^{x^2+1}$$

$f(0)=0$

15. $f: \mathbb{R} \to \mathbb{R}$ παρ/μη

$$f'(x)\left(e^{f(x)} + e^{-f(x)}\right) = 2$$

ψάχνω τύπο $f(x)$

$$f'(x)\, e^{f(x)} + f'(x)\, e^{-f(x)} = 2$$

$$\left(e^{f(x)} - e^{-f(x)}\right)' = (2x)'$$

$$e^{f(x)} - e^{-f(x)} = 2x + c$$

$\underline{x=0}$

$$e^{f(0)} - e^{-f(0)} = 0 + c$$

$$1 - 1 = c \qquad c = 0$$

$$e^{f(x)} - e^{-f(x)} = 2x$$

$$e^{2f(x)} - 1 = 2x e^{f(x)}$$

$$e^{2f(x)} - 2x e^{f(x)} + x^2 = x^2 + 1$$

$$\left(e^{f(x)} - x\right)^2 = \sqrt{x^2+1}^{\,2}$$

$$\left| e^{f(x)} - x \right| = \left| \sqrt{x^2+1} \right|$$

$$|h(x)| = \sqrt{x^2+1}$$

Ρίζα $h(x)$

$h(x) = 0$

$|h(x)| = 0$

$\sqrt{x^2+1} = 0$

Αδύνατη

$h(x) \neq 0$ και συνεχής

$h(x) > 0$ ή $h(x) < 0$

$h(0) = e^{f(0)} - 0 = 1$

$h(x) > 0$

$h(x) = \sqrt{x^2+1}$

$e^{f(x)} - x = \sqrt{x^2+1}$

$e^{f(x)} = \sqrt{x^2+1} + x$

$f(x) = \ln\left(\sqrt{x^2+1} + x\right)$

16. $f: \mathbb{R} \to \mathbb{R}$ και F παράγουσα $\quad F(0)=0$

$$(F(x)-e^x)(f(x)-e^x) = 2x^3+2x$$

Ψάχνω τύπο $f(x)$

$$2(F(x)-e^x)(f(x)-e^x) = 4x^3+4x$$

$$\left[(F(x)-e^x)^2\right]' = (x^4+2x^2)'$$

$$(F(x)-e^x)^2 = x^4+2x^2+c$$

$$\xrightarrow{x=0}$$

$$(F(0)-1)^2 = 0+0+c$$

$$1 = c$$

$$(F(x)-e^x)^2 = x^4+2x^2+1$$

$$(F(x)-e^x)^2 = (x^2+1)^2$$

$$|F(x)-e^x| = |\overset{+}{x^2+1}|$$

$$|h(x)| = x^2+1$$

Ρίζα $h(x)$

$h(x)=0$

$|h(x)|=0$

$x^2+1=0$

Άτοπο!

$h(x)\neq 0$ και συνεχής

$h(x)>0$ ή $h(x)<0$

$h(0)=F(0)-e^0=-1$

$h(0)=-1$

$h(x)<0$

$$|\overset{\ominus}{h(x)}|=x^2+1$$

$$-h(x)=x^2+1$$

$$-(F(x)-e^x)=x^2+1$$

$$F(x)-e^x=-x^2-1$$

$$F(x)=e^x-x^2-1$$

$$F'(x)=e^x-2x$$

$$\boxed{f(x)=e^x-2x}$$

17. $f:R \to R$ $\quad f(0)=1 \quad f'(0)=0$

$f(x)>0 \quad f(x)\left(f''(x)+2f(x)\right)=\left(f'(x)\right)^2$

Βρείτε $f(x)$

$$f(x)f''(x)+2f^2(x)=\left(f'(x)\right)^2$$

$$\frac{f(x)f''(x)-\left(f'(x)\right)^2}{f^2(x)}=-2$$

$$\left(\frac{f'(x)}{f(x)}\right)'=(-2x)'$$

$$\frac{f'(x)}{f(x)}=-2x+c$$

$\underline{x=0}$

$$\frac{f'(0)}{f(0)}=0+c$$

$$0=c$$

$$\frac{f'(x)}{f(x)}=-2x$$

$$\left(\ln f(x)\right)'=\left(-x^2\right)'$$

$$\ln f(x)=-x^2+c$$

$\underline{x=0}$

$\ln f(0)=c \quad c=0$

$\ln f(x)=-x^2 \quad f(x)=e^{-x^2}$

18. $f: \mathbb{R} \to \mathbb{R}$ [illegible] $f(x) > 0$

$f(1) = e^2$

$f'(x) = 2x f(x)$

$f'(x) - 2x\, f(x) = 0$

$g(x) = -2x$

$G(x) = -x^2$

$e^{G(x)} = e^{-x^2}$

$e^{-x^2} f'(x) - 2x\, e^{-x^2} = 0$

$\left(e^{-x^2} f(x)\right)' = 0$

$e^{-x^2} f(x) = c$

$\underline{x=1}$

$e^{-1} f(1) = c$

$e^{-1} e^{2} = c$

$c = e$

$e^{-x^2} f(x) = e$

$f(x) = e \cdot e^{x^2}$ $\qquad \underline{f(x) = e^{x^2+1}}$

Γραμμική Διαφορική Εξίσωση

$$f'(x) + g(x) f(x) = 0$$

Βρίσκω παράγουσα $G(x)$ της $g(x)$

Πολλ/ζω παντού με $e^{G(x)}$

$$e^{G(x)} f'(x) + g(x) e^{G(x)} f(x) = 0$$

$$\left(e^{G(x)} f(x)\right)' = 0$$

$$e^{G(x)} f(x) = c$$

19. $2f(x) + 4x f'(x) + (x^2+9)f''(x) = 0$

$f(0) = 0 \qquad f'(0) = \frac{1}{9}$

Νδο $f(x) = \dfrac{x}{x^2+9}$

$f: R \to R$ δυο φορές παρ/μη.

$\underbrace{(x^2+9) f(x) - x}_{g(x)} = 0$

$g'(x) = 2x f(x) + (x^2+9) f'(x) - 1$

$g''(x) = 2f(x) + 2x f'(x) + 2x f'(x) + (x^2+9) f''(x)$

$g''(x) = 2f(x) + 4x f'(x) + (x^2+9) f''(x) = 0$

$g''(x) = 0$

$g'(x) = c$

$2f(x) + (x^2+9) f'(x) - 1 = c$

$\underline{x=0}$

$2f(0) + 9f'(0) - 1 = c$

$9\frac{1}{9} - 1 = c \qquad c = 0$

$$g'(x) = 0$$

$$g(x) = c$$

$$(x^2+9)f(x) - x = c$$

$$\frac{x=0}{c=0}$$

$$f(x) = \frac{x}{x^2+9}$$

20. $f: \mathbb{R} \to \mathbb{R}$ συνεχής.

$$f(x) = 12x^2 - 2x \int_0^1 f(t)\,dt$$

Ψάχνω τύπο $f(x)$

Το $\int_0^1 f(t)\,dt$ είναι ορισμένο ολοκλήρωμα

και είναι αριθμός

Θέτω $\int_0^1 f(t)\,dt = k$

$$f(x) = 12x^2 - 2k \cdot x$$

$$\int_0^1 f(x)\,dx = \int_0^1 12x^2 - 2kx\,dx$$

$$k = 12\int_0^1 x^2\,dx - 2k\int_0^1 x\,dx$$

$$k = 12\,\frac{1}{3}\left(x^3\right)_0^1 - 2k\,\frac{1}{2}\left(x^2\right)_0^1$$

$$k = 4 - k$$

$$2k = 4 \qquad \underline{k = 2}$$