Θεματικη Ενοτητα

Αντιστροφη

1. Έστω $f(x) = \ln\left(\frac{x}{1-x}\right)$, $x \in (0,1)$

$$f'(x) = \frac{1}{\frac{x}{1-x}} \cdot \left(\frac{x}{1-x}\right)'$$

$$f'(x) = \left(\frac{1-x}{x}\right) \cdot \frac{1-x+x}{(1-x)^2} = \frac{(1-x)}{x} \frac{1}{(1-x)^2}$$

$$f'(x) = \frac{1}{x(1-x)} > 0 \Rightarrow f \nearrow$$

$\oplus \ \oplus$ στο $(0,1)$

$\Sigma T_f = D_{f^{-1}}$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \ln\left(\frac{x}{1-x}\right) = -\infty$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \ln\left(\frac{x}{1-x}\right) = \ln +\infty = +\infty$$

$$\rightarrow \lim_{x \to 1^-} \frac{x}{1-x} = \lim_{x \to 1^-} x \cdot \frac{1}{1-x} = 1 \cdot (+\infty) = +\infty$$

$D_{f^{-1}} = \mathbb{R}$.

Θετω $f(x)=y$

$$y=\ln\left(\frac{x}{1-x}\right)$$

$$e^{y}=\frac{x}{1-x}$$

$$e^{y}(1-x)=x$$

$$e^{y}-e^{y}x=x$$

$$e^{y}=e^{y}x+x$$

$$e^{y}=(e^{y}+1)x$$

$$x=\frac{e^{y}}{e^{y}+1}$$

$$f^{-1}(x)=\frac{e^{x}}{e^{x}+1}$$

$$D_{f^{-1}}=\mathbb{R}.$$

Προσοχη

Εαν η συναρτηση δινεται στις

$$f(x) = \ln x - \ln(1-x)$$

για να βρω την αντιστροφη πρεπει

$$f(x) = \ln\left(\frac{x}{1-x}\right)$$

2. Έστω $f(x) = (x-1)^2$, $x \in [0,1]$

$f'(x) = 2(x-1) \leq 0 \Rightarrow f \downarrow$.

$\left.\begin{array}{l} f(0) = 1 \\ f(1) = 0 \end{array}\right\} \Sigma T_f = D_{f^{-1}} = [0,1]$

Θέτω $y = f(x)$

$y = (x-1)^2$

$\sqrt{y}^2 = (x-1)^2$

$|\overset{\oplus}{\sqrt{y}}| = |\overset{\ominus}{x-1}|$

$\sqrt{y} = 1 - x$

$x = 1 - \sqrt{y}$

$$\boxed{\begin{array}{l} f^{-1}(x) = 1 - \sqrt{x} \\ D_{f^{-1}} = [0,1]. \end{array}}$$

3. $f(x) = x^3$

$f'(x) = 3x^2 \geq 0 \Rightarrow f \nearrow$

$$\left.\begin{array}{l} \lim\limits_{x \to -\infty} f(x) = -\infty \\ \lim\limits_{x \to +\infty} f(x) = +\infty \end{array}\right\} \quad \Sigma T_f = \mathbb{R}. \quad D_{f^{-1}} = \mathbb{R}.$$

Θέτω $f(x) = y$

$y = x^3$

$$x = \begin{cases} -\sqrt[3]{-y}, & y < 0 \\ \sqrt[3]{y}, & y \geq 0 \end{cases}$$

$$f^{-1}(x) = \begin{cases} -\sqrt[3]{-x}, & x < 0 \\ \sqrt[3]{x}, & x \geq 0 \end{cases}$$

4. $$f(x) = \begin{cases} x+2, & x \leq 0 \\ x^2+3, & x > 0 \end{cases}$$

$\underline{x < 0}$

$f_1(x) = x+2$

$f_1'(x) = 1 > 0$

$f_1 \nearrow$

$\underline{x > 0}$

$f_2(x) = x^2+3$

$f_2'(x) = 2x > 0$

$f_2 \nearrow$

Άρα $f \nearrow$

x		0	
f_1'	+		//////
f_2'	////		+
f'	+		+
f	$-\infty \nearrow 2$		$3 \nearrow +\infty$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$\Sigma\ T_f = D_{f^{-1}} = \mathbb{R}.$

$\underline{x<0}$

$f(x)=y$

$y=x+2$

$x=y-2$

$f^{-1}(x)=x-2$

$\underline{x>0}$

$f(x)=y$

$y=x^2+3$

$y-3=x^2$

$\sqrt{y-3}^{\,2}=x^2$

$|\overset{+}{\sqrt{y-3}}|=|\overset{+}{x}|$

$\sqrt{y-3}=x$

$f^{-1}(x)=\sqrt{x-3}$

$$f^{-1}=\begin{cases} x-2, & x\leq 2 \\ \sqrt{x-3}, & x>3 \end{cases}$$

5. Έστω $f(x) = \ln\left(\sqrt{x^2+1} - x\right)$, $x>0$

$$f'(x) = \frac{1}{\sqrt{x^2+1} - x} \cdot \left(\frac{2x}{2\sqrt{x^2+1}} - 1\right)$$

$$f'(x) = \frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1} - x} \cdot \frac{1}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{x^2+1}} < 0$$

$f \downarrow$.

$f(x) = y$

$$y = \ln\left(\sqrt{x^2+1} - x\right)$$

$$e^y = \sqrt{x^2+1} - x$$

$$e^y + x = \sqrt{x^2+1}$$

$$\left(e^y + x\right)^2 = x^2+1$$

$$e^{2y} + 2xe^y + \cancel{x^2} = \cancel{x^2} + 1$$

$$2xe^y = 1 - e^{2y}$$

$$x = \frac{1-e^{2y}}{2e^y} \qquad f^{-1}(x) = \frac{1-e^{2x}}{2e^x}$$

6. Έστω $f:\mathbb{R}\to\mathbb{R}$ παραγωγίσιμη.

$$f^3(x)+f(x)=x+1$$

Αφού f παραγωγίσιμη

$$3f^2(x)f'(x)+f'(x)=1$$

$$f'(x)\left(3f^2(x)+1\right)=1$$

$$f'(x)=\frac{1}{3f^2(x)+1}>0 \quad f\nearrow$$

$\boxed{\text{Θέτω } f(x)=y \text{ και } f^{-1}(y)=x}$

$$y^3+y=f^{-1}(y)+1$$

$$f^{-1}(y)=y^3+y-1$$

$$\underline{f^{-1}(x)=x^3+x-1}$$

7. Έστω $f(x) = x^3 + 2x + 1$, $x \in \mathbb{R}$.

$f'(x) = 3x^2 + 2 > 0$ $\quad f \nearrow$

Εδώ δεν μπορώ να βρω αντίστροφη!

Βρείτε E: $C_{f^{-1}}$, $x = 1$, $x = -2$, $x'x$

$$E = \int_{-2}^{1} |f^{-1}(x)|\, dx \overset{(*)}{=}$$

$$\boxed{\begin{array}{ll} \text{Θέτω} & f^{-1}(x) = t \\ & f(f^{-1}(x)) = f(t) \\ & x = f(t) \\ & dx = f'(t)\,dt. \\ x = 1 & t = 0 \\ x = -2 & t = -1 \end{array}}$$

$$\underline{\underline{(*)}} \quad \int_{-1}^{0} |\overset{\ominus}{t}|\, f'(t)\, dt = \int_{-1}^{0} -t(3t^2+2)\,dt$$

$$= \int_{-1}^{0} -3t^3 - 2t\, dt = \int_{-1}^{0} -3t^3 dt - \int_{-1}^{0} 2t\, dt$$

$$= -\frac{3}{4}\left(t^4\right)_{-1}^{0} - \left(t^2\right)_{-1}^{0} = \frac{3}{4} + 1 = \frac{7}{4}.$$

Βρεί την εφαπτομένη της $f^{-1}(x)$

στο $A(1, f^{-1}(1))$

$$y - f^{-1}(1) = f^{-1\prime}(1)(x-1)$$

Αφού $f(0)=1$ το $\underline{\underline{f^{-1}(1)=0}}$

Γνωρίζω ότι $f(f^{-1}(x)) = x$

$$f'(f^{-1}(x)) \cdot f^{-1\prime}(x) = 1$$

$$\underline{x=1}$$

$$f'(f^{-1}(1))\ f^{-1\prime}(1) = 1$$

$$f'(0)\ f^{-1\prime}(1) = 1$$

$$2 \cdot f^{-1\prime}(1) = 1$$

$$f^{-1\prime}(1) = \frac{1}{2}$$

$$y - 0 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x - \frac{1}{2}$$

8. Δινεται $f(x) = x^3+2x^2+2x$

$f'(x) = 3x^2+4x+2$

$\Delta = 16-24<0$

$f'(x)>0 \Rightarrow f \nearrow$

Βρει τα κοινα σημεια των $C_{f^{-1}}$, $y=x$

Δεν μπορω να βρω την αντιστροφη!

$f^{-1}(x) = x$

$f(f^{-1}(x)) = f(x)$

$x = f(x)$

$x = x^3+2x^2+2x$

$0 = x^3+2x^2+x$

$0 = x(x^2+2x+1)$

$0 = x(x+1)^2$

$x=0$ $\quad$ $x=-1$

$A(0,0)$ $\quad$ $B(-1,-1)$

Βρες τα κοινά σημεία των $C_{f^{-1}}, C_f$

$f^{-1}(x) = f(x)$

$\Leftrightarrow$

$f(x) = x$

$x=0$ $\quad$ $x=-1$

$A(0,0)$ $\quad$ $B(-1,-2)$

$f^{-1}(x) = f(x)$

$f(f^{-1}(x)) = f(f(x))$

$x = f(f(x))$

$f(x) + x = f(f(x)) + f(x)$

$\boxed{g(x) = f(x) + x}$

$g(x) = g(f(x))$

g 1-1

$f(x) = x$

Μονοτονία
$g(x)$

- $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- $x_1 < x_2$

$\oplus$

$g(x_1) < g(x_2)$

$g \nearrow$

$E : C_f , C_{f^{-1}}$

$$E = \int_{-1}^{0} | f(x) - f^{-1}(x) | \, dx \ \underline{\underline{(*)}}$$

- $f(x) = f^{-1}(x) \iff x = 0 \ , x = -1$

$$\underline{\underline{(*)}} \int_{-1}^{0} 2 \, | f(x) - x | \, dx =$$

$$= 2 \int_{-1}^{0} | x^3 + 2x^2 + 2x - x | \, dx =$$

$$= 2 \int_{-1}^{0} | x^3 + 2x^2 + x | \, dx =$$

$$= 2 \int_{-1}^{0} | x \, (x^2 + 2x + 1) | \, dx$$

$$= 2 \int_{-1}^{0} | \overset{\ominus}{x} \, \overset{\oplus}{(x+1)^2} | \, dx$$

$$= 2 \int_{-1}^{0} - x \, (x+1)^2 \, dx$$

$$= 2\int_{-1}^{0} -x^3 - 2x^2 - x \quad dx$$

$$= -\frac{2}{4}(x^4)_{-1}^{0} - \frac{4}{3}(x^3)_{-1}^{0} - \frac{2}{2}(x^2)_{-1}^{0}$$

$$= \frac{2}{4} + \frac{4}{3} + \frac{2}{2} = \frac{6}{12} + \frac{16}{12} + \frac{12}{12} = \frac{34}{12}.$$

9. Εστω οτι η f ειναι γνησιως αυξουσα.

Να βρειτε τη μονοτονια της $f^{-1}(x)$

Εστω $f^{-1}(x_1) < f^{-1}(x_2)$

$$f \nearrow$$

$$f(f^{-1}(x_1)) < f(f^{-1}(x_2))$$

$$x_1 < x_2$$

$$f^{-1} \nearrow$$