Θεματικη Ενοτητα

Ανισοτητες

1. Έστω $f(x) = e^x + x + συνx$, $x \in R$.

$$f'(x) = e^x + \underbrace{1 - ημx}_{\oplus} > 0 \qquad f \nearrow$$

Ν.δ.ο $f(εφx) - e^x > x + συνx \quad \forall x \in (0, \frac{\pi}{2})$

$$f(εφx) > e^x + x + συνx$$

$$f(εφx) > f(x)$$

$$f \nearrow$$

$$εφx > x \implies εφx - x > 0$$

$\boxed{\text{Θεωρώ } g(x) = εφx - x}$

$$g'(x) = \frac{1}{συν^2x} - 1 = \frac{1 - συν^2x}{συν^2x} = \frac{ημ^2x}{συν^2x} \geq 0$$

$$g \nearrow$$

Αν $x > 0 \implies g(x) > g(0)$

$$εφx - x > 0$$

$$εφx > x$$

Νδο $f(e^{x-1}-1) > f(\ln x^{x}) \quad \forall x>1$

Αφού $f \nearrow$

$$e^{x-1}-1 > \ln x^{x}$$

$$e^{x-1}-1 > x\ln x$$

$$e^{x-1}-1-x\ln x > 0$$

$\boxed{\text{Θεωρώ } g(x) = e^{x-1}-1-x\ln x}$

$g'(x) = e^{x-1} - \ln x - 1 \qquad g'(1)=0$

$g''(x) = e^{x-1} - \frac{1}{x} \qquad g''(1)=0$

$g'''(x) = e^{x-1} + \frac{1}{x^2} > 0$

x		1	
g'''	+		+
g''	$\nearrow$ −	0	+ $\nearrow$
g'	$\searrow$ +	0	+ $\nearrow$
g	$\nearrow$		$\nearrow$

Αν $x>1 \Rightarrow g(x) > g(1)$

$$e^{x-1}-1-x\ln x > 0$$

2.

Νδο $x^2 \geq 2\ln x + 1$, $x>0$

$$x^2 - 2\ln x - 1 \geq 0$$

$$\boxed{\text{Θεωρω } g(x) = x^2 - 2\ln x - 1}$$

$$g'(x) = 2x - \frac{2}{x} = \frac{2x^2 - 2}{x} = \frac{2(x^2-1)}{x}$$

$\longrightarrow g'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow \underline{\underline{x = 1}}$

x	0	1	$+\infty$
g'	$-$	0	$+$
g	$\searrow$		$\nearrow$

$$g(x) \geq g(1)$$

$$x^2 - 2\ln x - 1 \geq 0$$

3

Ν.δ.ο. $e^{-x} \geq 1-x$

Γνωρίζω $e^{x} \geq x+1 \quad \forall x \in \mathbb{R}$

άρα $e^{-x} \geq -x+1$

$e^{-x} \geq 1-x$

Ν.δ.ο. $\ln(x^2+1) \leq x^2 \quad \forall x \in \mathbb{R}$

Γνωρίζω ότι $\ln x \leq x-1 \quad \forall x > 0$

$\ln(x^2+1) \leq x^2+1-1$

$\ln(x^2+1) \leq x^2$

Ν.δ.ο. $\ln\left(1+\frac{1}{x}\right) < \frac{1}{x} \quad \forall x > 0$

Γνωρίζω $\ln x \leq x-1$

$\ln\left(1+\frac{1}{x}\right) \leq \frac{1}{x}+1-1 \Rightarrow \ln\left(1+\frac{1}{x}\right) \leq \frac{1}{x}$

Ν.δ.ο. $\ln(x+1) \geq \frac{x}{x+1} \quad \forall x > 0$

Γνωρίζω ότι $\ln x \leq x-1$ άρα $\ln\frac{1}{x+1} \leq \frac{1}{x+1}-1$

$\ln 1 - \ln(x+1) \leq \frac{1}{x+1} - \frac{x+1}{x+1} \Leftrightarrow -\ln(x+1) \leq -\frac{x}{x+1}$

$\ln(x+1) \geq \frac{x}{x+1}.$

4

Έστω ότι η f κυρτή

Νδο $f'(x) > \dfrac{f(x)}{x-2}$ $\forall x>2$ αν $f(2)=0$

$$f'(\xi) = \frac{f(x)-f(2)}{x-2} = \frac{f(x)-0}{x-2}$$

2 ξ x

$$\xi < x \xrightarrow[f \text{ κυρτή}]{f'\uparrow} f'(\xi) < f'(x) \Rightarrow \frac{f(x)}{x-2} < f'(x)$$

Νδο $f'(x) < f(x+1) - f(x)$ $\forall x \in \mathbb{R}$

$$f'(\xi) = \frac{f(x+1)-f(x)}{x+1-x} = f(x+1)-f(x)$$

x ξ x+1

$$x < \xi \xrightarrow{f'\uparrow} f'(x) < f'(\xi)$$

$$f'(x) < f(x+1) - f(x)$$

Νδο $f(4x) - f(2x) < 2x\, f'(4x)$ $\forall x>0$

$$f'(\xi) = \frac{f(4x)-f(2x)}{4x-2x} = \frac{f(4x)-f(2x)}{2x}$$

2x ξ 4x

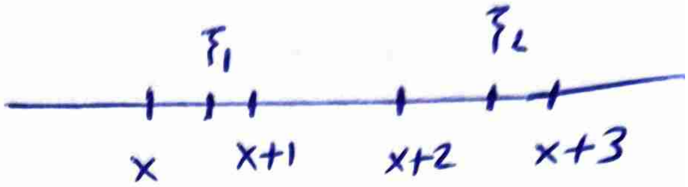
$$\xi < 4x \Rightarrow f'(\xi) < f'(4x)$$

$$\frac{f(4x)-f(2x)}{2x} < f'(4x)$$

$$f(4x)-f(2x) < 2x\, f'(4x)$$

Νδο $f(x+1) + f(x+2) < f(x) + f(x+3) \quad \forall x \in R.$

$$f'(\xi_1) = \frac{f(x+1) - f(x)}{x+1-x} = f(x+1) - f(x)$$

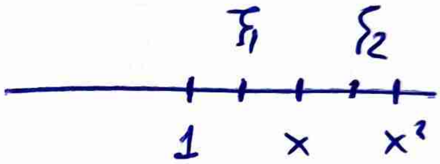


$$f'(\xi_2) = \frac{f(x+3) - f(x+2)}{x+3-x-2} = f(x+3) - f(x+2)$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2) \Rightarrow f(x+1) - f(x) < f(x+3) - f(x+2)$$

$$f(x+1) + f(x+2) < f(x) + f(x+3)$$

Νδο $(x+1)f(x) < x f(1) + f(x^2) \quad \forall x > 1$



$$f'(\xi_1) = \frac{f(x) - f(1)}{x-1}$$

$$f'(\xi_2) = \frac{f(x^2) - f(x)}{x^2 - x}$$

$$\xi_1 < \xi_2 \Rightarrow f'(\xi_1) < f'(\xi_2)$$

$$\frac{f(x) - f(1)}{\cancel{x-1}} < \frac{f(x^2) - f(x)}{x\cancel{(x-1)}}$$

$$x f(x) - x f(1) < f(x^2) - f(x)$$

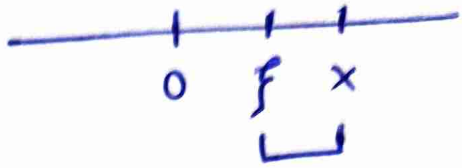
$$x f(x) + f(x) < f(x^2) + x f(1)$$

$$(x+1) f(x) < f(x^2) + x f(1).$$

Αν $f'(x) > 1$ Νδο $x < f(x) < x f'(x)$ $\forall x > 0$

και $f(0)=0$

$$f'(\xi) = \frac{f(x) - f(0)}{x - 0}$$



$$f'(\xi) = \frac{f(x)}{x}$$

$$\xi < x \Rightarrow f'(\xi) < f'(x) \Rightarrow \frac{f(x)}{x} < f'(x)$$

$$\underline{f(x) < x f'(x)}$$

Γνωριζω οτι

$$f'(x) > 1$$

$$f'(\xi) > 1$$

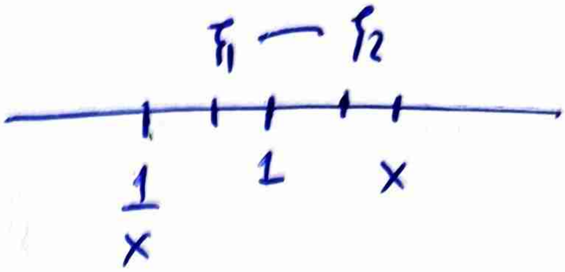
$$\frac{f(x)}{x} > 1$$

$$\underline{f(x) > x}$$

Μεγαλη προσοχη οταν μου δινουν ανισοτητα για την $f'(x)$

Αν $f'(x)>0$ νδο $(x+1)\,G(1) < G(x) + G\left(\frac{1}{x}\right)x$

αν $x>1$ και $G(x)$ παράγουσα της $f(x)$



$$G'(\xi_1) = \frac{G(1)-G\left(\frac{1}{x}\right)}{1-\frac{1}{x}}$$

$$G'(\xi_2) = \frac{G(x)-G(1)}{x-1}$$

$$\xi_1 < \xi_2 \xRightarrow{G'\nearrow} G'(\xi_1) < G'(\xi_2)$$

$$\boxed{\begin{array}{l}\text{Αφού } f'(x)>0 \\ \Rightarrow G''(x)>0 \\ \quad G'\nearrow\end{array}}$$

$$\frac{G(1)-G\left(\frac{1}{x}\right)}{1-\frac{1}{x}} < \frac{G(x)-G(1)}{x-1}$$

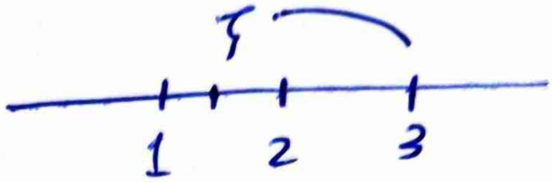
$$x\,\frac{G(1)-G\left(\frac{1}{x}\right)}{x-1} < \frac{G(x)-G(1)}{x-1}$$

$$x\,G(1) - x\,G\left(\frac{1}{x}\right) < G(x) - G(1)$$

$$x\,G(1) + G(1) < x\,G\left(\frac{1}{x}\right) + G(x)$$

$$(x+1)\,G(1) < x\,G\left(\frac{1}{x}\right) + G(x)$$

Αν $f'(x) > 0$ νδο $G(2) < f(3) + G(1)$



$$G'(\xi) = \frac{G(2)-G(1)}{2-1} = G(2)-G(1)$$

$$\xi < 3 \overset{G'\nearrow}{\Rightarrow} G'(\xi) < G'(3)$$

$$G(2)-G(1) < f(3)$$

$$G(2) < f(3) + G(1)$$

5.

Έστω $f(x) = \ln x - \frac{1}{x}$, $x > 0$

Ν.δ.ο. $f(x) \leq 2x - 3$, $\forall x > 0$

$$f'(x) = \frac{1}{x} + \frac{1}{x^2}$$

$$f''(x) = -\frac{1}{x^2} - \frac{2}{x^3} < 0$$ f κοίλη

$$\left.\begin{aligned} y - f(1) &= f'(1)(x-1) \\ y - (-1) &= 2(x-1) \\ y + 1 &= 2x - 2 \\ y &= 2x - 3 \end{aligned}\right\}$$

Αφού f κοίλη $f(x) \leq 2x - 3$

Ν.δ.ο. $f(x^2+1) + 1 \leq 2x^2$

Αφού $f(x) \leq 2x - 3$ $\forall x > 0$

$$f(x^2+1) \leq 2(x^2+1) - 3$$

$$f(x^2+1) \leq 2x^2 - 1$$

$$f(x^2+1) + 1 \leq 2x^2$$

Νδο $\int_1^4 f(x)\sqrt{x}\,dx < \frac{54}{5}$

Γνωρίζω ότι $f(x) \le 2x-3$

$$f(x)\sqrt{x} \le (2x-3)\sqrt{x}$$

$$\int_1^4 f(x)\sqrt{x}\,dx < \int_1^4 (2x-3)\sqrt{x}\,dx$$

$$\longrightarrow \int_1^4 (2x-3)\sqrt{x}\,dx \xrightarrow[x=t^2]{\sqrt{x}=t} \int_1^2 (2t^2-3)\,t\,2t\,dt$$

$$dx = 2t\,dt$$

$$= 2\int_1^2 (2t^2-3)t^2\,dt = 2\int_1^2 2t^4-3t^2\,dt =$$

$$= 4\int_1^2 t^4\,dt - 6\int_1^2 t^2\,dt =$$

$$= \frac{4}{5}\left(t^5\right)_1^2 - \frac{6}{3}\left(t^3\right)_1^2$$

$$= \frac{4\cdot 31}{5} - 2\cdot 7 = \frac{124}{5} - 14 = \frac{124}{5} - \frac{70}{5}$$

$$= \frac{54}{5}$$

6.

Έστω $f: \mathbb{R} \to \mathbb{R}$ συνεχής
και $f \nearrow$ και $f(0)=0$ και $f(1)=1$

Ν.δ.ο. $0 < \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu x \, f(\eta\mu x)\, dx < 1$

$$\rightarrow \int_0^{\frac{\pi}{2}} \sigma\upsilon\nu x \, f(\eta\mu x)\, dx \;\xrightarrow[\sigma\upsilon\nu x\, dx = dt]{\eta\mu x = t}\; \int_0^1 f(t)\, dt$$

Άρα ν.δ.ο. $0 < \int_0^1 f(x)\, dx < 1$

Είναι $0 < x < 1$

$$f \nearrow$$

$$f(0) < f(x) < f(1)$$

$$0 < f(x) < 1$$

$$\int_0^1 0\, dx < \int_0^1 f(x)\, dx < \int_0^1 1\, dx$$

$$0 < \int_0^1 f(x)\, dx < (x)_0^1$$

$$0 < \int_0^1 f(x)\, dx < 1.$$

Νδο $\int_0^1 f(e^x)\,dx < 2$ αν $f(e)=2$

- $0<x<1 \Rightarrow e^0 < e^x < e^1 \Rightarrow 1 < e^x < e$

$f(1) < f(e^x) < f(e)$

$1 < f(e^x) < 2$

αρα αφου $f(e^x) < 2$

$\int_0^1 f(e^x)\,dx < \int_0^1 2\,dx$

$\int_0^1 f(e^x)\,dx < 2.$

Β' τροπος

$\rightarrow \int_0^1 f(e^x)\,dx \overset{e^x=t}{=\!=} \int_1^e \frac{f(t)}{t}\,dt = \int_1^e \frac{f(x)}{x}\,dx$

$e^x dx = dt$

$t\,dx = dt$

$dx = \frac{1}{t}\,dt$

Αρκει νδο $\int_1^e \frac{f(x)}{x}\,dx < 2$

Οταν $1<x<e \Rightarrow f(1) < f(x) < f(e)$

$1 < f(x) < 2$

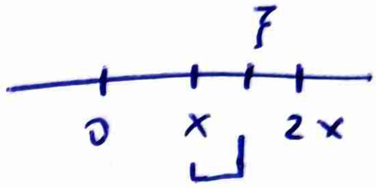
$\frac{1}{x} < \frac{f(x)}{x} < \frac{2}{x}$

$\Rightarrow \int_1^e \frac{f(x)}{x}\,dx < \int_1^e \frac{2}{x}\,dx \Rightarrow \int_1^e \frac{f(x)}{x}\,dx < 2.$

7.

Έστω $f: [0,+\infty) \to \mathbb{R}$ συνεχής και κυρτή.

Νδο $f(2x) - f(x) \geq x f'(x) \quad \forall x \geq 0$



$$f'(\xi) = \frac{f(2x)-f(x)}{2x-x} = \frac{f(2x)-f(x)}{x}$$

$$x < \xi \overset{f \text{ κυρτή}}{\underset{f' \uparrow}{\Longrightarrow}} f'(x) < f'(\xi) \Rightarrow f'(x) < \frac{f(2x)-f(x)}{x}$$

$$x f'(x) < f(2x) - f(x)$$

Νδο $\int_0^1 f(2x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx$

$$\to \int_0^1 f(2x)\,dx \underset{2dx=dt}{\overset{2x=t}{=\!=}} \int_0^2 f(t)\frac{1}{2}\,dt = \frac{1}{2}\int_0^2 f(x)\,dx$$

Νδο $\int_0^2 f(x)\,dx > 2f(1)$

$$f(2x) - f(x) \geq x f'(x)$$

$$\Rightarrow \int_0^1 f(2x) - f(x)\,dx > \int_0^1 x f'(x)\,dx$$

$$\int_0^1 f(2x)\,dx - \int_0^1 f(x)\,dx > \int_0^1 x f'(x)\,dx$$

$$\frac{1}{2}\int_0^2 f(x)\,dx > \int_0^1 x f'(x)\,dx + \int_0^1 f(x)\,dx$$

$$\frac{1}{2}\int_0^2 f(x)\,dx > \int_0^1 \left(x f'(x) + f(x)\right) dx$$

$$\frac{1}{2}\int_0^2 f(x)\,dx > \left(x f(x)\right)_0^1$$

$$\frac{1}{2}\int_0^2 f(x)\,dx > f(1)$$

$$\int_0^2 f(x)\,dx > 2 f(1).$$

8

Δινεται $f(x) = \frac{e^x}{x}$, $x > 0$

ΝΔΟ $\int_{\frac{1}{2}}^{2} f(x)\,dx > \frac{3e}{2}$

$$f'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

x	0	1	
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

$f(x) \geq f(1)$

$f(x) \geq e$

Αφου $f(x) \geq e$

$$\int_{\frac{1}{2}}^{2} f(x)\,dx > \int_{\frac{1}{2}}^{e} e\,dx$$

$$\int_{\frac{1}{2}}^{2} f(x)\,dx > e\,(x)_{\frac{1}{2}}^{2}$$

$$\int_{\frac{1}{2}}^{2} f(x)\,dx > e\left(2 - \frac{1}{2}\right)$$

$$\int_{\frac{1}{2}}^{e} f(x)\,dx > \frac{3e}{2}$$

9

Δίνεται $f(x) = \ln(x^2+1)$

ⓐ Ν.δ.ο. $f(x) \leq x^2 \quad \forall x \in \mathbb{R}$.

Ισχύει ότι $\ln x \leq x-1$ άρα και $\ln(x^2+1) \leq x^2+1-1$

$\Leftrightarrow \ln(x^2+1) \leq x^2 \Leftrightarrow f(x) \leq x^2$.

ⓑ Ν.δ.ο. $\int_0^1 f(x)\,dx < \frac{1}{3}$

Αφού $f(x) \leq x^2$

$$\int_0^1 f(x)\,dx \leq \int_0^1 x^2\,dx$$

$$\int_0^1 f(x)\,dx \leq \frac{1}{3}.$$

10

ⓐ Ν.δ.ο $\sqrt{x+1} \le \frac{x+2}{2}$ $\quad \forall x \ge -1$

$$\boxed{f(x) = \sqrt{x+1} - \frac{x+2}{2}}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2} = \frac{1-\sqrt{x+1}}{2\sqrt{x+1}}$$

- $1-\sqrt{x+1} = 0$

$1 = \sqrt{x+1}$

$1 = x+1$

$x = 0$

x	-1	0	
f'	$+$		$-$
f	↗		↘

$f(x) \le f(0)$

$\sqrt{x+1} - \frac{x+2}{2} \le 0$

$\sqrt{x+1} \le \frac{x+2}{2}$

Ⓑ Ν.δ.ο $\int_0^1 \sqrt{x^3+1}\, dx \le \frac{9}{8}$

Ισχυει οτι $\sqrt{x^3+1} \le \frac{x^3+2}{2}$

$$\int_0^1 \sqrt{x^3+1}\, dx \le \int_0^1 \frac{x^3+2}{2}\, dx$$

$$\int_0^1 \sqrt{x^3+1}\, dx \le \frac{9}{8}.$$

11

Εστω $f: (0,+\infty) \longrightarrow \mathbb{R}$ με $f(1)=1$ ωστε

$$x f'(x) + 1 < x e^{x-1}, \ \forall x>0$$

Ν.δ.ο $f(x) < e^{x-1} - \ln x \quad \forall x > 1$.

$$f(x) < e^{x-1} - \ln x \iff f(x) - e^{x-1} + \ln x < 0$$

$$\boxed{g(x) = f(x) - e^{x-1} + \ln x}$$

$$g'(x) = f'(x) - e^{x-1} + \frac{1}{x} = \frac{\overset{\ominus}{x f'(x) - x e^{x-1} + 1}}{x} < 0$$

$\Rightarrow g \downarrow$.

Αρα αν $x>1 \Rightarrow g(x) < g(1) \iff$

$$f(x) - e^{x-1} + \ln x < 0$$

$$f(x) < e^{x-1} - \ln x.$$

12

Εστω $f: R \longrightarrow R$ με $f(0)=0$ παραγωγισιμη.

και ισχυει $f'(x) > 2\left(x f(x) + e^{x^2}\right) \quad \forall x \in R$.

(α) νδο $f(x) \geq 2x e^{x^2} \quad \forall x \in [0,1]$.

Ισχυει οτι $f'(x) > 2x f(x) + 2e^{x^2}$

$$f'(x) - 2x f(x) > 2e^{x^2}$$

$\bullet\ g(x) = -2x$

$\bullet\ G(x) = -x^2$

$e^{G(x)} = e^{-x^2}$

$$e^{-x^2} f'(x) - 2x e^{-x^2} f(x) > 2e^{x^2} e^{-x^2}$$

$$\left(e^{-x^2} f(x)\right)' > 2.$$

$$\Leftrightarrow \left[e^{-x^2} f(x) - 2x\right]' > 0.$$

Θεωρω $g(x) = e^{-x^2} f(x) - 2x$

η οποια ειναι ↗ αφου $g'(x) > 0$

Αρα αν $x \geq 0 \Rightarrow g(x) \geq g(0)$

$$e^{-x^2} f(x) - 2x \geq 0$$

$$e^{-x^2} f(x) \geq 2x \quad \Leftrightarrow f(x) \geq 2x e^{x^2}$$

Ⓑ νδο $\int_0^1 f(x)\,dx > e-1$.

Αφού $f(x) \geq 2xe^{x^2} \rightarrow \int_0^1 f(x)\,dx > \int_0^1 2xe^{x^2}dx$

$\Leftrightarrow \int_0^1 f(x)\,dx > \left(e^{x^2}\right)_0^1 \Leftrightarrow \int_0^1 f(x)\,dx > e-1$.

Ⓖ Νδο η εξίσωση $x\int_0^1 f(x)\,dx = e-x$

εχει ακριβως μια ριζα στο $(0,1)$.

$\boxed{\text{Θεωρω } \varphi(x) = x\int_0^1 f(x)\,dx - e + x}$

$\varphi(0) = -e$

$\varphi(1) = \int_0^1 f(x)\,dx - e + 1 > 0$

Bolzano $\exists \xi \in (0,1)$ τ.ω $\varphi(\xi) = 0$

$\varphi'(x) = \underbrace{\int_0^1 f(x)\,dx}_{\oplus} + 1 > 0 \Rightarrow \varphi \nearrow$

γιατι $\int_0^1 f(x)\,dx > e-1$

13

Δινεται $f(x)= 2e^x - x^2 - x - 1$. $D_f = \mathbb{R}$

(α) Να μελετηθει ως προς την κυρτοτητα και τα σημεια καμπης.

$f'(x) = 2e^x - 2x - 1$

$f''(x) = 2e^x - 2$.

$\longrightarrow 2e^x - 2 = 0 \Leftrightarrow 2e^x = 2 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$.

x		0	
f''	$-$	0	$+$
f	$\cap$		$\cup$

(β) ΝΔΟ $f(x) \geq x+1$, $\forall x \geq 0$.

Η εφαπτομενη στο $x_0=0$ ειναι $y - f(0) = f'(0)(x-0)$

$y - 1 = 1\cdot(x-0)$

$y - 1 = x$

$\forall x \geq 0$ η f κυρτη αρα

$f(x) \geq x+1$.

(8) i) Ν.δ.ο $\int_1^e \frac{f(x)}{x}dx > e$

Γνωρίζω ότι $\forall x \geq 0$ $f(x) \geq x+1 \Leftrightarrow \frac{f(x)}{x} \geq 1+\frac{1}{x}$

$$\int_1^e \frac{f(x)}{x}dx \geq \int_1^e 1+\frac{1}{x}\,dx \Leftrightarrow \int_1^e \frac{f(x)}{x}dx > e.$$

$$\rightarrow \int_1^e 1+\frac{1}{x}\,dx = (x+\ln x)_1^e = e+\ln e-(1+\ln 1) =$$
$$= e+1-1 = e$$

ii). Ν.δ.ο $\int_1^2 x f(x)\,dx > \frac{23}{6}$

Γνωρίζω ότι $\forall x \geq 0$ $f(x) \geq x+1 \Leftrightarrow x f(x) \geq x^2+x$

$$\int_1^2 x f(x)\,dx \geq \int_1^2 x^2+x\,dx \Leftrightarrow \int_1^2 x f(x)\,dx > \frac{23}{6}.$$

$$\rightarrow \int_1^2 x^2+x\,dx = \int_1^2 x^2 dx + \int_1^2 x\,dx = \frac{1}{3}(x^3)_1^2 + \frac{1}{2}(x^2)_1^2$$
$$= \frac{7}{3}+\frac{3}{2} = \frac{14}{6}+\frac{9}{6} = \frac{23}{6}$$

iii). Ν.δ.ο $\int_0^1 x f(e^x)\,dx > \frac{3}{2}$

Γνωρίζω ότι $\forall x \geq 0$ $f(x) \geq x+1 \Rightarrow f(e^x) \geq e^x+1$

$$\Leftrightarrow x f(e^x) \geq x e^x + x \Leftrightarrow \int_0^1 x f(e^x)\,dx > \int_0^1 x e^x + x\,dx$$

$$\rightarrow \int_0^1 x e^x + x\,dx = \int_0^1 x(e^x)'dx + \int_0^1 x\,dx = (xe^x)_0^1 - \int_0^1 e^x dx + \frac{1}{2}(x^2)_0^1$$
$$= e-(e^x)_0^1 + \frac{1}{2} = e-(e-1)+\frac{1}{2} = e-e+1+\frac{1}{1} = \frac{3}{2}$$

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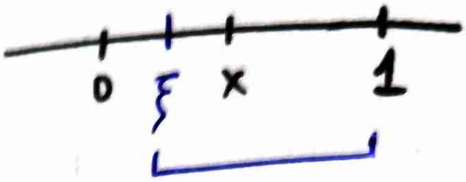
Εστω $f:\mathbb{R}\to\mathbb{R}$ με $f(0)=0$, κυρτη με $f'(1)=3$

(α) Νδο $f(x)\le 3x \quad \forall x\in[0,1]$

εφαρμοζω ΘΜΤ για την $f(x)$ στο $[0,x]$



$$f'(\xi)=\frac{f(x)-f(0)}{x-0}=\frac{f(x)}{x}$$

ομως

$$\xi<1 \overset{f'\uparrow}{\Rightarrow} f'(\xi)<f'(1) \Leftrightarrow \frac{f(x)}{x}<3 \Leftrightarrow f(x)<3x.$$

(β) Νδο $\exists\, a\in(0,1)$ μοναδικο τ.ω. $a\int_0^1 f(x^2)\,dx=2a-1$.

$$\boxed{\text{Θεωρω } g(x)=x\int_0^1 f(t^2)\,dt-2x+1.}$$

$g(0)=1$

$g(1)=\int_0^1 f(t^2)\,dt-1<0$

Αφου $f(x)\le 3x \Leftrightarrow f(x^2)\le 3x^2 \Leftrightarrow \int_0^1 f(x^2)\,dx\le\int_0^1 3x^2\,dx$

$\Leftrightarrow \int_0^1 f(x^2)\,dx \le (x^3)_0^1$

$\int_0^1 f(x^2)\,dx\le 1 \Rightarrow \int_0^1 f(x^2)\,dx-1\le 0.$

Το $g(x)$ πολυωνυμο 1ου βαθμου, η ριζα μοναδικη.

15

Δινεται συνεχης συναρτηση $f(x)=\begin{cases}\dfrac{\ln x}{x-1}, & 0<x\neq 1\\[2mm] 1, & x=1\end{cases}$

Εστω α σταθερος αριθμος

οπου $a\in(0,1)$

Νδο $\displaystyle\int_1^{a} f(ax)\,dx < \frac{1}{a^2}\int_a^{a^2} x\,f(x)\,dx$

Λοιπον $\displaystyle\int_1^{a} f(ax)\,dx \;\overset{ax=t}{\underset{a\,dx=dt}{=\!=}}\; \int_a^{a^2} f(t)\,\frac{1}{a}\,dt =$

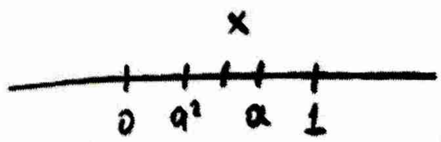
$\displaystyle = \frac{1}{a}\int_a^{a^2} f(t)\,dt.$

Συνεπως αρκει νδο $\displaystyle\frac{1}{a}\int_a^{a^2} f(x)\,dx < \frac{1}{a^2}\int_a^{a^2} x f(x)\,dx$

$\displaystyle \Leftrightarrow \int_{a^2}^{a} a\,f(x)\,dx > \int_{a^2}^{a} x\,f(x)\,dx$

Δηλαδη αρκει νδο $a f(x) > x f(x)$

$\Leftrightarrow \underset{+}{(a-x)}\,\underset{+}{f(x)} > 0$



Οταν $0<x<1$ τοτε $\left.\begin{matrix}x-1<0\\ \ln x<0\end{matrix}\right\}\ \dfrac{\ln x}{x-1}>0$

16

Εστω $f: R \to R$ συνεχη και γνησια φθινουσα

Νδο $\int_0^1 f(x)\,dx > 2\int_{\frac{1}{2}}^1 f(2x)\,dx$

$\to \int_{1/2}^1 f(2x)\,dx \underset{\substack{2dx=dt \\ dx=\frac{1}{2}dt}}{\overset{2x=t}{=\!=}} \int_1^2 \frac{1}{2} f(t)\,dt =$

$= \frac{1}{2}\int_1^2 f(x)\,dx$

Αρκει νδο $\int_0^1 f(x)\,dx > 2\,\frac{1}{2}\int_1^2 f(x)\,dx$

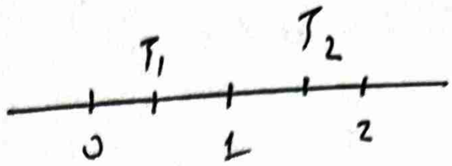
Αφου f συνεχη εχει παραγουσα $F(x)$

$$\boxed{\int_0^1 f(x)\,dx > \int_1^2 f(x)\,dx}$$

$\left(F(x)\right)_0^1 > \left(F(x)\right)_1^2$

$F(1) - F(0) > F(2) - F(1)$

$$F'(\xi_1) = \frac{F(1)-F(0)}{1-0} = F(1)-F(0)$$



$$F'(\xi_2) = \frac{F(2)-F(1)}{2-1} = F(2)-F(1)$$

$$\xi_1 < \xi_2 \xRightarrow[f\downarrow]{F'\downarrow} F'(\xi_1) > F'(\xi_2)$$

$$F(1)-F(0) > F(2)-F(1)$$